Subelliptic pseudo-differential operators on compact Lie groups

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Abstract

In this work we extend the theory of global pseudo-differential operators on compact Lie groups to a subelliptic context. More precisely, given a compact Lie group G, and the sub-Laplacian $\mathcal{L}$ associated to a system of vector fields $X = \{X_1, \cdots, X_k\}$ satisfying the Hörmander condition, we introduce a (subelliptic) pseudo-differential calculus associated to $\mathcal{L}$, based on the matrix-valued quantisation process developed in [2].

Introduction

- In modern mathematics, the theory of pseudo-differential operators is a powerful branch in the analysis of linear partial differential operators due to its interactions with several areas of mathematics.
- For instance, from the point of view of the theory of PDE, pseudo-differential operators are used e.g.
  - I. to study the global/local solvability of several partial differential problems.
  - II. To understand the mapping properties of certain singular integral operators.
  - III. To understand the propagation of singularities in distribution theory, and in the construction of fundamental solutions and parametrices.
  - IV. To compute some geometric invariants arising in the index theory.
- Here we describe how in [1], we associate to every sub-Riemannian structure of a compact Lie group a pseudo-differential calculus.

Compact Lie groups

- $$\text{Compact Lie Group} = \begin{cases} \text{Closed manifold} \\ + \\ \text{topological group} \end{cases}$$
- Examples: $\mathbb{T}^n$, $\mathrm{SU}(n)$, $\mathrm{Spin}(n)$.

Fourier analysis on compact Lie groups

- If G is a compact Lie group, its unitary dual $\widehat{G}$ consists of all equivalence classes of continuous irreducible unitary representations of G.
- Unitary Representation: for each ξ from the equivalence class $[\xi] \in \widehat{G}$ we have
$$\xi \in \mathrm{HOM}(G, U(H_\xi))$$
for some (finite-dimensional) vector space $H_\xi \cong \mathbb{C}^{d_\xi}$. We denote by $d_\xi = dim(H_\xi)$ the dimension of the representation ξ.

$$\widehat{f}(\xi) := \int_G f(x)\xi(x)^* dx, \quad \xi \in [\xi] \in \widehat{G}.$$

Pseudo-differential operators on compact Lie groups

- Every continuous linear operator T from $C^\infty(G)$ to itself admits a representation in the following (quantization) formula:

$$Tf(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\xi(x)a(x,\xi)\widehat{f}(\xi)],$$

- The matrix-valued function a defined on the phase space $G \times \widehat{G}$ is called the symbol of a. For every ξ, $a(x,\xi) := \xi(x)^*(A\xi)(x)$ is a matrix of size $d_\xi \times d_\xi$.

Laplacian vs sub-Laplacian

- The positive sub-Laplacian $\mathcal{L}$ associated to a system of vector fields $X = \{X_1, \cdots, X_k\}$ satisfying the Hörmander condition, is defined by $\mathcal{L} = -X_1^2 - \cdots - X_k^2$.
- If $X' := \{X_1, \cdots, X_n\}$ is a basis of the Lie algebra $\mathfrak{g}$, of G, the positive Laplacian $\mathcal{L}_G$ is defined by $\mathcal{L} = -X_1^2 - \cdots - X_n^2$.

Subelliptic pseudo-differential calculus

The Laplacian generates an elliptic pseudo-differential calculus on every compact Lie group [2]. The elliptic Hörmander classes for this calculus are denoted by $\mathscr{S}^m_{\rho,\delta}(G\times\widehat{G})$. The theory can be extended to sub-Riemannian structures induced by any choice of a sub-Laplacian [1]. The subelliptic classes are denoted by $S^{m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$.

Main results and applications of the theory

- The subelliptic pseudo-differential calculus is closed under compositions, inverses, parametrices, complex powers, etc.
- (Boundedness properties)
  - I. (Fefferman estimates). Let Q be the Hausdorff dimension of G associated to the control distance associated to the sub-Laplacian $\mathcal{L}$. If $m \geqslant m_p := Q(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|$, then
  $$\mathrm{Op}(S^{-m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})) \subset \mathscr{B}(L^p(G))\, 1 < p < \infty.$$
  - II. (Calderón-Vaillancourt Theorem) For $0 \leqslant \delta \leqslant \rho \leqslant 1/2\kappa$, or $0 \leq \delta < \rho \leq 1$,
  $$\mathrm{Op}(S^{0,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})) \subset \mathscr{B}(L^2(G))\, 1 < p < \infty.$$
- (Gårding Inequality)
$$\mathrm{Re}(a(x,D)u,u) \geqslant C_1\|u\|_{L^{2,\mathcal{L}}_{\frac{m}{2}}(G)} - C_2\|u\|^2_{L^2(G)}.$$
- Well posedness for the Cauchy problem
$$(\mathrm{PVI}) : \begin{cases} \frac{\partial v}{\partial t} = K(t,x,D)v + f, \\ v(0) = u_0, v \in \mathscr{D}'((0,T)\times G) \end{cases} \quad (1)$$
- Asymptotic expansions in spectral geometry
$$\mathbf{Tr}(A\psi(tE)) = t^{-\frac{Q+m}{q}}\left(\sum_{k=0}^{\infty} a_k t^k\right) + \frac{c_Q}{q}\int_0^\infty \psi(s)\times\frac{ds}{s},$$
- L^p-boundedness of Fourier integral operators.
- Dixmier traces, classification in Schatten-von Neumann classes.
- Sharp-L^p-estimates for oscillatory Fourier multipliers.

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