Some new results on q-Dunkl harmonic analysis

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(Joint work with **Othman Tyr**)

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Dedicate

Dedicated to the memory of my dear daughter Joumana-Daher

At just 15 years of age a beautiful soul in Joumana-Daher (called also VAEDEHI) passed on May 26, 2020.

An Epilog

This is joint work with my PhD student, Mr. Othman Tyr of University Hassan II in Casablanca. In this short talk, I will describe some results in his PhD dissertation. I have chosen:

- q-Analog of Titchmarsh's theorems [8].
- q-Version of equivalence theorem between a K-functional and modulus of smoothness [9].
- q-Approximation theory "direct and inverse theorems for Jackson" [10].

Summary

Introduction

- **Lipschitz condition**

$$\mathbb{L}ip_{\mathbb{R}}(\alpha, p) = \{f \in L^p(\mathbb{R}) : \|\tau_h f - f\|_p = O(h^\alpha), \text{as } h \to 0\}$$

where $\tau_h f = f(. + h)$ is the usual translation and $0 < \alpha \leq 1$.

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Theorem A (E.C. Titchmarsh, Theorem 84)

Let $0 < \alpha \leq 1$ and assume that $f \in L^p(\mathbb{R})$. If $f \in \mathbb{L}ip_{\mathbb{R}}(\alpha, p)$, then its Fourier transform $\widehat{f}$ belong to $L^\beta(\mathbb{R})$ for

$$\frac{p}{p + \alpha p - 1} < \beta \leq \frac{p}{p - 1}.$$

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Theorem B (E.C. Titchmarsh, Theorem 85)

Let $\alpha \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then, the following statement are equivalents:

(i) $f \in \mathbb{L}ip_{\mathbb{R}}(\alpha, 2)$,

(ii) $$\int_{|\lambda|\geq N} |\widehat{f}(\lambda)|^2 d\lambda = O(N^{-2\alpha}) \quad \text{as} \quad N \to \infty,$$

where $\widehat{f}$ stands for the classical Fourier transform of f.

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where

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Theorem C (P.L. Butzer and H. Behrens 1967)

There are two positive constants C_1 and C_2 such that for all $f \in L^2(\mathbb{R})$ and $\delta > 0$, we have

$$C_1\omega_m(f,\delta) \leq K_m(f,\delta^m) \leq C_2\omega_m(f,\delta).$$

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where $K_m(f,\delta)$ is the classical K-functional introduced in 1963 by J. Peetre, defined by

$$K_m(f,\delta) = \inf\{\|f-g\|_2 + \delta\|D^m g\|_2;\ g \in \mathcal{W}_2^m\},$$

with $\mathcal{W}_2^m$ be the Sobolev space constructed by the operator $D = \frac{d}{dx}$.

Our objective

The aim of this talk is to extend these results: "Titchmarsh's theorems 84 and 85", " Jackson's direct and inverse theorems" and "Equivalence theorem between a K-functional and modulus of smoothness" to the q-harmonic analysis associated with the q-Dunkl operator introduced by N. Bettaibi and al. in 2007.

Throughout this presentation we will fix $0 < q < 1$.

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- For all $a \in \mathbb{C}$, the q-Pochhammer symbols, also called the q-shifted factorials are defined by:

$$(a;q)_0 = 1, \ \ (a;q)_n = \prod_{l=0}^{n-1}(1 - aq^l), \ \ n = 1, 2, ...$$

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$$[x]_q = \frac{1-q^x}{1-q}, \ \ x \in \mathbb{C}, \quad [n]_q! = \prod_{k=1}^{n}[k]_q = \frac{(q,q)_n}{(1-q)^n}, \quad n \in \mathbb{N}.$$

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The q-gamma function is given by

$$\Gamma_q(x) = \frac{(q,q)_\infty}{(q^x,q)_\infty}(1-q)^{1-x}, \quad x \neq 0, -1, -2, ...$$

- The q-Jackson integrals are defined by

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{+\infty} q^n f(aq^n),$$

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- The q-derivatives $\mathcal{D}_q f$ and $\mathcal{D}_q^+ f$, are also known as the Jackson derivatives are defined by

$$\mathcal{D}_q f(x) = \frac{f(x)-f(qx)}{(1-q)x},\quad \mathcal{D}_q^+ f(x) = \frac{f(q^{-1}x)-f(x)}{(1-q)x}\quad \text{if } x\neq 0,$$

$\mathcal{D}_q f(0) = f^{'}(0)$ and $\mathcal{D}_q^+ f(0) = q^{-1}f^{'}(0)$ provided $f^{'}(0)$ exists.

- The Rubin's q-differential operator is defined in by

$$\partial_q f(x) = \begin{cases} \dfrac{f(q^{-1}x) + f(-q^{-1}x) - f(qx) + f(-qx) - 2f(-x)}{2(1-q)x} & \text{if } x \neq 0, \\ \lim\limits_{x\to 0} \partial_q f(x), \qquad (\text{in } \mathbb{R}_q) & \text{if } x = 0. \end{cases}$$

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Remark that if f is differentiable at x, then $\lim\limits_{q \to 1} \partial_q f(x) = f^{'}(x)$.

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- We denote by $L^p_{q,\alpha}(\mathbb{R}_q)$, $p \in [1, +\infty]$, the set of all real functions on $\mathbb{R}_q$ for which

$$\|f\|_{q,p,\alpha} = \begin{cases} \left(\displaystyle\int_{-\infty}^{+\infty} |f(x)|^p |x|^{2\alpha+1} d_q x \right)^{1/p} < +\infty & \text{if } 1 \leq p < +\infty, \\ \operatorname*{ess\,sup}\limits_{x\in\mathbb{R}_q} |f(x)| < +\infty & \text{if } p = +\infty. \end{cases}$$

- For $\alpha \geq -\frac{1}{2}$, the q-Dunkl operator is defined by

$$\Lambda_{q,\alpha}(f)(x) = \partial_q[\mathcal{H}_{q,\alpha}(f)](x) + [2\alpha+1]_q \frac{f(x) - f(-x)}{2x},$$

where

$$\mathcal{H}_{q,\alpha} : f = f_e + f_o \mapsto f_e + q^{2\alpha+1} f_o.$$

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- Note that if $\alpha = -\frac{1}{2}$, $\Lambda_{q,\alpha} = \partial_q$.
- The q-differential-difference equation:

$$\Lambda_{q,\alpha}(f) = i\lambda f, \quad f(0) = 1.$$

has as unique solution, the function $\Psi_{q,\alpha}(\lambda.)$ defined by

$$\Psi_{q,\alpha}(\lambda x) = j_\alpha(\lambda x, q^2) + \frac{i\lambda x}{[2\alpha+2]_q} j_{\alpha+1}(\lambda x, q^2), \quad x \in \mathbb{R}_q,$$

where $j_\alpha(., q^2)$ is the normalized third Jackson's q-Bessel function given by

$$j_\alpha(x, q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma_{q^2}(\alpha+1) q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)} \left(\frac{x}{1+q}\right)^{2n}.$$

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The function $\Psi_{q,\alpha}(\lambda.)$ admits the following properties.

Theorem 1 (N. Bettaibi, 2007)

i) For all $\lambda, x \in \mathbb{R}$, $\overline{\Psi_{q,\alpha}(\lambda x)} = \Psi_{q,\alpha}(-\lambda x)$.

ii) For all $\lambda, x \in \mathbb{R}_q$, $\Lambda_{q,\alpha}\Psi_{q,\alpha}(\lambda x) = i\lambda\Psi_{q,\alpha}(\lambda x)$.

iii) If $\alpha = -\frac{1}{2}$, then $\Psi_{q,\alpha}(\lambda x) = e(i\lambda x, q^2)$.

iv) For all $\lambda \in \mathbb{R}_q$, $\Psi_{q,\alpha}(\lambda.)$ is bounded on $\widetilde{\mathbb{R}}_q$ and we have

$$|\Psi_{q,\alpha}(\lambda x)| \leq \frac{4}{(q,q)_\infty}, \ \ \forall x \in \widetilde{\mathbb{R}}_q.$$

Definition 1 (N. Bettaibi 2007)

The q-Dunkl transform $\mathcal{F}_D^{q,\alpha}$ is defined on $L^1_{q,\alpha}(\mathbb{R}_q)$ by

$$\mathcal{F}_D^{q,\alpha}(f)(\lambda) = c_{q,\alpha} \int_{-\infty}^{\infty} f(x)\Psi_{q,\alpha}(-\lambda x)|x|^{2\alpha+1} d_q x, \quad \forall \lambda \in \mathbb{R}_q,$$

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where
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It satisfies the following properties:

- If $\alpha = -1/2$, $\mathcal{F}_D^{q,\alpha}$ is the q^2-analogue Fourier transform $\widehat{f}(.,q^2)$ given by

$$\widehat{f}(\lambda,q^2) = \frac{(1+q)^{\frac{1}{2}}}{2\Gamma_{q^2}(\frac{1}{2})}\int_{-\infty}^{\infty} f(x)e(-i\lambda x,q^2)d_qx.$$

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$$\mathcal{F}_D^{q,\alpha}(f)(\lambda) = c_{q,\alpha}\int_{-\infty}^{\infty} f(x)\Psi_{q,\alpha}(-\lambda x)|x|^{2\alpha+1}d_qx, \quad \forall\lambda \in \mathbb{R}_q,$$

where
$$c_{q,\alpha} = \frac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)}.$$

It satisfies the following properties:

- If $\alpha = -1/2$, $\mathcal{F}_D^{q,\alpha}$ is the q^2-analogue Fourier transform $\widehat{f}(., q^2)$ given by

$$\widehat{f}(\lambda, q^2) = \frac{(1+q)^{\frac{1}{2}}}{2\Gamma_{q^2}(\frac{1}{2})}\int_{-\infty}^{\infty} f(x)e(-i\lambda x, q^2)d_qx.$$

- On the even functions space, $\mathcal{F}_D^{q,\alpha}$ coincides with the q-Bessel transform

- **$L^1 - L^\infty$-boundedness:**
  For all $f \in L^1_{q,\alpha}(\mathbb{R}_q)$, we have $\mathcal{F}_D^{q,\alpha}(f) \in L^\infty_{q,\alpha}(\mathbb{R}_q)$ and

$$\|\mathcal{F}_D^{q,\alpha}(f)\|_{q,\infty} \leq \frac{4c_{q,\alpha}}{(q,q)_\infty}\|f\|_{q,1,\alpha}.$$

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- **Riemann-Lebesque Lemma:**
  For all $f \in L^1_{q,\alpha}(\mathbb{R}_q)$, we have

$$\lim_{\substack{|\lambda|\to\infty \\ \lambda\in\mathbb{R}_q}} \mathcal{F}_D^{q,\alpha}(f)(\lambda) = 0.$$

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- **q-Plancherel formula:**
  The q-Dunkl transform $\mathcal{F}_D^{q,\alpha}$ is an isomorphism from $L^2_{q,\alpha}(\mathbb{R}_q)$ (resp. $\mathcal{S}_q(\mathbb{R}_q)$) onto itself and satisfies the following q-Plancherel formula:
  $$\|\mathcal{F}_D^{q,\alpha}(f)\|_{q,2,\alpha} = \|f\|_{q,2,\alpha} \quad \text{for all } \ f \in L^2_{q,\alpha}(\mathbb{R}_q).$$

- **q-Inversion formula:**
  Let f be a function in $L^1_{q,\alpha}(\mathbb{R}_q)$, such that $\mathcal{F}_D^{q,\alpha}(f)$ belongs to $L^1_{q,\alpha}(\mathbb{R}_q)$. Then

$$f(x) = c_{q,\alpha}\int_{-\infty}^{\infty}\mathcal{F}_D^{q,\alpha}(f)(\lambda)\Psi_{q,\alpha}(\lambda x)|\lambda|^{2\alpha+1}d_q\lambda.$$

- **q-Inversion formula:**
  Let f be a function in $L^1_{q,\alpha}(\mathbb{R}_q)$, such that $\mathcal{F}^{q,\alpha}_D(f)$ belongs to $L^1_{q,\alpha}(\mathbb{R}_q)$. Then

$$f(x) = c_{q,\alpha}\int_{-\infty}^{\infty}\mathcal{F}^{q,\alpha}_D(f)(\lambda)\Psi_{q,\alpha}(\lambda x)|\lambda|^{2\alpha+1}d_q\lambda.$$

Proposition 1 (q-Hausdorff Young inequality)

Let $f \in L^p_{q,\alpha}(\mathbb{R}_q)$, $p \geq 1$, then $\mathcal{F}^{q,\alpha}_D(f) \in L^{p'}_{q,\alpha}(\mathbb{R}_q)$. If $1 \leq p \leq 2$, then

$$\|\mathcal{F}^{q,\alpha}_D(f)\|_{q,p',\alpha} \leq C_p\|f\|_{q,p,\alpha},$$

where

$$C_p = \left(\frac{4c_{q,\alpha}}{(q,q)_\infty}\right)^{\frac{2}{p}-1}$$

is a positive constant and the numbers p and p' above are conjugate exponents:

Definition 2

The generalized q-Dunkl translation operator is defined for $f \in L^2_{q,\alpha}(\mathbb{R}_q)$ and $x, h \in \mathbb{R}_q$ by

$$T_h^{q,\alpha}(f)(x) = c_{q,\alpha} \int_{-\infty}^{\infty} \mathcal{F}_D^{q,\alpha}(f)(\lambda)\Psi_{q,\alpha}(\lambda x)\Psi_{q,\alpha}(\lambda h)|\lambda|^{2\alpha+1} d_q\lambda,$$

$$T_0^{q,\alpha}(f) = f.$$

It satisfies the following properties.

Theorem 2 (N. Bettaibi 2010)

i) For all $x, h \in \mathbb{R}_q$, we have $T_h^{q,\alpha}(f)(x) = T_x^{q,\alpha}(f)(h)$.

ii) If $f \in L^2_{q,\alpha}(\mathbb{R}_q)$, (resp. $\mathcal{S}_q(\mathbb{R}_q)$) then $T_h^{q,\alpha}(f) \in L^2_{q,\alpha}(\mathbb{R}_q)$ (resp. $\mathcal{S}_q(\mathbb{R}_q)$) and we have

$$\|T_h^{q,\alpha}(f)\|_{q,2,\alpha} \leq \frac{4}{(q,q)_\infty}\|f\|_{q,2,\alpha}, \quad \forall h \in \mathbb{R}_q.$$

iii) For all $x, h, \lambda \in \mathbb{R}_q$, we have

$$T_h^{q,\alpha}(\Psi_{q,\alpha}(\lambda.))(x) = \Psi_{q,\alpha}(\lambda x)\Psi_{q,\alpha}(\lambda h).$$

iv) For $f \in L^2_{q,\alpha}(\mathbb{R}_q)$ and $x, h \in \mathbb{R}_q$, we have

$$\mathcal{F}_D^{q,\alpha}(T_h^{q,\alpha}f)(\lambda) = \Psi_{q,\alpha}(\lambda h)\mathcal{F}_D^{q,\alpha}(f)(\lambda).$$

Lemma 1:

The following inequalities are fulfilled:

i) For all $x \in \mathbb{R}_q$, there exist a constant $C > 0$ such that

$$|1 - \Psi_{q,\alpha}(x)| \leq C|x|.$$

ii) The inequality

$$|1 - \Psi_{q,\alpha}(x)| \geq c$$

is true with $x \geq 1$, $x \in \mathbb{R}_q$, where $c > 0$ is a certain constant.

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is true with $x \geq 1$, $x \in \mathbb{R}_q$, where $c > 0$ is a certain constant.

Definition 3:

Let $0 < \delta < 1$. A function $f \in L^p_{q,\alpha}(\mathbb{R}_q)$, $p \geq 1$ is said to be in the q-Dunkl-Lipschitz class, denoted by q-$\mathcal{D}Lip(\delta, p, \alpha)$, if

$$\| T_h^{q,\alpha} f - f\|_{q,p,\alpha} = O(h^\delta), \quad \text{as } h \to 0.$$

- q-version of Theorem A (Titchmarsh's theorem 84)

Theorem 1: (R. Daher and O. Tyr 2020)

Let f belongs to $L^p_{q,\alpha}(\mathbb{R}_q)$, $1 < p \leq 2$ and let f also belongs to q-$\mathcal{DL}ip(\delta, p, \alpha)$. Then $\mathcal{F}^{q,\alpha}_D(f)$ belongs to $L^\beta_{q,\alpha}(\mathbb{R}_q)$, where

$$\frac{2p\alpha + 2p}{2p + 2\alpha(p-1) + \delta p - 2} < \beta \leq p' = \frac{p}{p-1}.$$

- q-version of Theorem A (Titchmarsh's theorem 84)

Theorem 1: (R. Daher and O. Tyr 2020)

Let f belongs to $L^p_{q,\alpha}(\mathbb{R}_q)$, $1 < p \leq 2$ and let f also belongs to q-$\mathcal{DL}ip(\delta, p, \alpha)$. Then $\mathcal{F}_D^{q,\alpha}(f)$ belongs to $L^\beta_{q,\alpha}(\mathbb{R}_q)$, where

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- q-version of Theorem B (Titchmarsh's theorem 85)

Theorem 2: (R. Daher and O. Tyr 2020)

Let $0 < \delta < 1$ and assume that $f \in L^2_{q,\alpha}(\mathbb{R}_q)$. Then the following statement are equivalents:

(1) $f \in$ q-$\mathcal{D}Lip(\delta, 2, \alpha)$.

(2) $\displaystyle\int_{|\lambda| \geq r} |\mathcal{F}_D^{q,\alpha}(f)(\lambda)|^2 |\lambda|^{2\alpha+1} d_q\lambda = O(r^{-2\delta}), \quad \text{as} \quad r \to \infty.$

- We define the generalized modulus of smoothness $\omega_m^{(\alpha)}(f,\delta)_{q,2}$ of order m in the space $L^2_{q,\alpha}(\mathbb{R}_q)$ by the formula

$$\omega_m^{(\alpha)}(f,\delta)_{q,2} = \sup_{0<h\leq\delta} \|\Delta_h^m f\|_{q,2,\alpha}, \;\; \delta > 0, f \in L^2_{q,\alpha}(\mathbb{R}_q),$$

where

$$\Delta_h^m f(x) = (T_h^{q,\alpha} - I)^m f(x).$$

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where

$$\Delta_h^m f(x) = (T_h^{q,\alpha} - I)^m f(x).$$

- The Sobolev space $\mathcal{W}^m_{q,2,\alpha}(\mathbb{R}_q)$ constructed by $\Lambda_{q,\alpha}$ is defined by

$$\mathcal{W}^m_{q,2,\alpha}(\mathbb{R}_q) := \{f \in L^2_{q,\alpha}(\mathbb{R}_q) : \; \Lambda^j_{q,\alpha} f \in L^2_{q,\alpha}(\mathbb{R}_q), \;\; j = 1, 2, ..., m\},$$

where

$$\Lambda^0_{q,\alpha} f = f, \quad \Lambda^j_{q,\alpha} f = \Lambda_{q,\alpha}(\Lambda^{j-1}_{q,\alpha} f), \quad j = 1, 2, ..., m.$$

Definition 1:

A function $f \in L^2_{q,\alpha}(\mathbb{R}_q)$ is called a function with bounded spectrum of order $\sigma > 0$ if $\mathcal{F}_D^{q,\alpha} f(\lambda) = 0$ for $|\lambda| > \sigma$. The set of all such functions is denoted by $\mathcal{I}^{(\alpha)}_{q,\sigma}$.

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Definition 2:

The best approximation of a function $f \in L^2_{q,\alpha}(\mathbb{R}_q)$ by functions in $\mathcal{I}^{(\alpha)}_{q,\sigma}$ is the quantity

$$E_\sigma(f)_{q,2,\alpha} := \inf_{g \in \mathcal{I}^{(\alpha)}_{q,\sigma}} \|f - g\|_{q,2,\alpha}.$$

- q-version of Jackson's direct theorem:

Theorem 1: (R. Daher and O. Tyr 2020)

Let $f \in L^2_{q,\alpha}(\mathbb{R}_q)$, $m \in \mathbb{N}$, then the following inequality holds for any $\sigma > 0$:

$$E_\sigma(f)_{q,2,\alpha} \leq c_1 \omega_m^{(\alpha)}(f, 1/\sigma)_{q,2}, \tag{1}$$

where c_1 is a positive constant.

- q-version of Jackson's direct theorem:

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where c_1 is a positive constant.

Theorem 2: (R. Daher and O. Tyr 2020)

Assume that $f, \Lambda_{q,\alpha} f, ..., \Lambda^d_{q,\alpha} f$, $d \in \mathbb{N}$, belong to $L^2_{q,\alpha}(\mathbb{R}_q)$, where $\Lambda_{q,\alpha}$ is the q-Dunkl operator. Then

$$E_\sigma(f)_{q,2,\alpha} \leq c_2 \sigma^{-d} \omega_m^{(\alpha)}(\Lambda^d_{q,\alpha} f, 1/\sigma)_{q,2},$$

where c_2 is a positive constant.

- q-version of Bernstein's Theorem

Theorem 3:

For $f \in \mathcal{I}_{\alpha}^{(q,\sigma)}$ and $s \in \mathbb{N}$, we have the inequality

$$\|\Lambda_{q,\alpha}^{s} f\|_{q,2,\alpha} \leq \sigma^{s} \|f\|_{q,2,\alpha}.$$

- q-version of Bernstein's Theorem

Theorem 3:

For $f \in \mathcal{I}_{\alpha}^{(q,\sigma)}$ and $s \in \mathbb{N}$, we have the inequality

$$\|\Lambda_{q,\alpha}^{s} f\|_{q,2,\alpha} \leq \sigma^{s} \|f\|_{q,2,\alpha}.$$

- q-version of Jackson's inverse theorems

Theorem 4: (R. Daher and O. Tyr 2020)

For every function $f \in L^2_{q,\alpha}(\mathbb{R}_q)$ and every $n \in \mathbb{N}^*$, we have

$$\omega_m\left(f, \frac{1}{n}\right)_{q,2,\alpha} \leq \frac{c_3}{n^m} \sum_{j=0}^{n} (j+1)^{m-1} E_j(f)_{q,2,\alpha},$$

where $c_3 = c_3(m, \alpha, q)$ is a positive constant.

Theorem 5: (R. Daher and O. Tyr 2020)

Suppose that $f \in L^2_{q,\alpha}(\mathbb{R}_q)$ and

$$\sum_{j=1}^{+\infty} j^{m-1} E_j(f)_{q,2,\alpha} < \infty.$$

Then $f \in W^m_{q,2,\alpha}(\mathbb{R}_q)$ and for every $n \in \mathbb{N}^*$ and we have

$$\omega_k \left(\Lambda^m_{q,\alpha} f, \frac{1}{n} \right)_{q,2,\alpha}$$
$$\leq K \left(\frac{1}{n^k} \sum_{j=0}^{n} (j+1)^{k+m-1} E_j(f)_{q,2,\alpha} + \sum_{j=n+1}^{+\infty} j^{m-1} E_j(f)_{q,2,\alpha} \right),$$

where $K = K(k, \alpha, q)$ is a positive constant.

Definition 1:

The K-functional constructed by the spaces $L^2_{q,\alpha}(\mathbb{R}_q)$ and $\mathcal{W}^m_{q,2,\alpha}(\mathbb{R}_q)$ is defined by

$$K\left(f, \delta, L^2_{q,\alpha}(\mathbb{R}_q), \mathcal{W}^m_{q,2,\alpha}(\mathbb{R}_q)\right)$$
$$= \inf\left\{\|f-g\|_{2,\alpha,\beta} + \delta\|\Lambda^m_{q,\alpha} g\|_{2,\alpha,\beta} : g \in \mathcal{W}^m_{q,2,\alpha}(\mathbb{R}_q)\right\},$$

where $f \in L^2_{q,\alpha}(\mathbb{R}_q)$ and $\delta > 0$.

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$$= \inf\left\{\|f-g\|_{2,\alpha,\beta} + \delta\|\Lambda^m_{q,\alpha}g\|_{2,\alpha,\beta} : g \in \mathcal{W}^m_{q,2,\alpha}(\mathbb{R}_q)\right\},$$

where $f \in L^2_{q,\alpha}(\mathbb{R}_q)$ and $\delta > 0$.

- For brevity, we denote

$$K^{(\alpha)}_m(f,\delta)_{q,2} = K\left(f, \delta, L^2_{q,\alpha}(\mathbb{R}_q), \mathcal{W}^m_{q,2,\alpha}(\mathbb{R}_q)\right).$$

Theorem 1: (R. Daher and O. Tyr 2020)

There are two positive constants C_1 and C_2 such that

$$C_1\omega_m^{(\alpha)}(f,\delta)_{q,2} \leq K_m^{(\alpha)}(f,\delta^m)_{q,2} \leq C_2\omega_m^{(\alpha)}(f,\delta)_{q,2},$$

for all $f \in L^2_{q,\alpha}(\mathbb{R}_q)$ and $\delta > 0$.

References I

[1] Bettaibi, N., Bettaieb, R.H.: *q*-Analogue of the Dunkl transform on the real line. Tamsui Oxford Journal of Mathematical Sciences. Vol. 25, No. 2, 117-205 (2007)

[2] Bettaibi, N., Bettaieb, R.H., Bouaziz, S.: Wavelet transform associated with the q-Dunkl operator, Tamsui Oxford Journal of Mathematical Sciences, Vol. 26, No. 1, 77-101 (2010)

[3] S. Bernstein : Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités, Commun. Soc. Math. Kharkow., Vol. 13, No. 2 (1912-1913), pp. 1-2.

[4] Daher, R. and El ouadih, S.: Some new estimates of approximation of functions by Fourier-Jacobi sums, Ser. Math. Inform., Vol. 31, No. 1, 1-10. (2016)

References II

[5] Daher, R., El ouadih, S. and El hamma, M.: Direct and inverse theorems of approximation theory in $L^2(\mathbb{R}^d, w_l(x)dx)$, Matematika, Vol. 33, No. 2, 177–189 (2017)

[6] Daher, R., El Hamma, M.: An analog of Titchmarsh's theorem for the generalized Dunkl transform. J. Pseudo-Differ. Oper. Appl. 7, 59–65 (2016).

[7] Daher, R., Delgado, J., Ruzhansky M.; Titchmarsh theorems for Fourier transforms of Hölder-Lipschitz functions on compact homogeneous manifolds. Monatsh Math, Vol. 189, 23-49 (2019)

[8] Daher, R., Tyr, O.: An analog of Titchmarsh's theorem for the q-Dunkl transform in the space $L_{2,q}(\mathbb{R}_q)$. J. Pseudo-Dier. Oper. Appl. (2020)

References III

[9] Daher, R., Tyr, O. : Modulus of smoothness and theorems concerning approximation in the space $L_{2,q}(\mathbb{R}_q)$ with power weight, Mediterranean Journal of Mathematics, (2020)

[10] Daher, R., Tyr, O. : On the Jackson-type inequalities in approximation theory connected to the q-Dunkl operators in the weighted space $L_{2,q}(\mathbb{R}_q)$ with power weight, submitted in 2020

[11] Ditzian, Z., Totik, V.: Moduli of Smoothness. Springer, Berlin (1987)

[12] D. Jackson : über die genauigkeit der annäherung stetiger funktionen durch ganze rationale Functionen gegebenen Grades und trigonometrische summen gegebener ordnung, Diss. Göttingen, (1911).

References IV

[13] Gasper, G., Rahman, M.: Basic Hypergeometric Series. Encyclopedia of Mathematics and its application. Vol 35, Cambridge Univ. Press, Cambridge, UK (1990)

[14] Nikol'skii, S.M.: A generalization of an inequality of S. N. Bernstein. Dokl. Akad. Nauk. SSSR 60(9), 1507-1510 (1948)

[15] Peetre, J.: A theory of interpolation of normed spaces. Notes de Universidade de Brasilia, Brasilia, Vol. 88 (1963)

[16] Rubin, R.L.: A q^2-Analogue Operator for q^2-analogue Fourier Analysis. J. Math. analys. App. 212, 571-582 (1997)

[17] Titchmarsh, E.C.: Introduction of the Theory of Fourier Integrals. Oxford University Press, Oxford (1937)

References V

[18] Timan, A.F.: Theory of Approximation of Functions of a Real Variable. Fizmatgiz, Moscow (1960).(English transl., Pergamon Press, Oxford-New York (1963))

[19] Weierstrass, K.: über die analytische darstellbarkeit sogenannter willkürlicher functionen einer reellen Veränderlichen, Sitzungsber. Akad. Berlin, 633-639 (1885)

Thank You for your attention