Classification problems in operator algebras

Noncommutative conference

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Operator algebras

We consider $*$-subalgebras $M \subset B(H)$, where the $*$-operation is the Hermitian adjoint.

- **Operator norm:**
  for $T \in B(H)$, we put $\|T\| = \sup\{\|T\xi\| \mid \xi \in H, \|\xi\| \leq 1\}$.

  C*-algebras: norm closed $*$-subalgebras of $B(H)$.

- **Weak topology:**
  $T_i \to T$ if and only if $\langle T_i\xi, \eta\rangle \to \langle T\xi, \eta\rangle$ for all $\xi, \eta \in H$.

  Von Neumann algebras: weakly closed $*$-subalgebras of $B(H)$.

$\rightsquigarrow$ Intimate connections to group theory, dynamical systems, quantum information theory, representation theory, ...

Commutative operator algebras

- Unital commutative C*-algebras are of the form $C(X)$ where X is compact Hausdorff.

  $\rightsquigarrow$ algebraic topology, K-theory, continuous dynamics, geometric group theory

- Commutative von Neumann algebras are of the form $L^\infty(X, \mu)$ where (X, μ) is a standard probability space.

  $\rightsquigarrow$ ergodic theory, measurable dynamics, measurable group theory

Discrete groups and operator algebras

Let G be a countable (discrete) group.

- Left regular unitary representation $\lambda : G \to \mathcal{U}(\ell^2(G)) : \lambda_g \delta_h = \delta_{gh}$.
- $\text{span}\{\lambda_g \mid g \in G\}$ is the **group algebra** $\mathbb{C}[G]$.
- Take the norm closure: (reduced) **group C*-algebra** $C_r^*(G)$.
- Take the weak closure: **group von Neumann algebra** $L(G)$.

We have $G \subset \mathbb{C}[G] \subset C_r^*(G) \subset L(G)$.

At each inclusion, information gets lost $\rightsquigarrow$ natural rigidity questions.

Open problems

- Kaplansky's conjectures for torsion-free groups G.
  - Unit conjecture: the only invertibles in $\mathbb{C}[G]$ are multiples of group elements λ_g.
  - Idempotent conjecture: 0 and 1 are the only idempotents in $\mathbb{C}[G]$.
  - Kadison-Kaplansky: 0 and 1 are the only idempotents in $C_r^*(G)$.
- Free group factor problem: is $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ if $n \neq m$?
- Connes' rigidity conjecture: $L(\mathrm{PSL}(n, \mathbb{Z})) \not\cong L(\mathrm{PSL}(m, \mathbb{Z}))$ if $3 \leq n < m$.
- Stronger form: if G has property (T) and $\pi : L(G) \to L(\Gamma)$ is a $*$-isomorphism, then $G \cong \Gamma$ and π is essentially given by such an isomorphism.

$\rightsquigarrow$ Structure and classification of operator algebras is highly nontrivial.

Operator algebras and group actions

Let G be a countable group.

Continuous dynamics and C*-algebras

An action $G \curvearrowright X$ of G by homeomorphisms of a compact Hausdorff space X gives rise to the C*-algebra $C(X) \rtimes_r G$.

Measurable dynamics and von Neumann algebras

An action $G \curvearrowright (X, \mu)$ of G by measure class preserving transformations of (X, μ) gives rise to a von Neumann algebra $L^\infty(X) \rtimes G$.

- These operator algebras contain $C(X)$, resp. $L^\infty(X)$, as subalgebras.
- They contain G as unitary elements $(u_g)_{g \in G}$.
- They encode the group action: $u_g \, F \, u_g^* = \alpha_g(F)$ where $(\alpha_g(F))(x) = F(g^{-1} \cdot x)$.

Amenable von Neumann algebras: full classification

Definition (von Neumann)

A countable group G is amenable if there exists a finitely additive probability measure m on the subsets of G such that $m(g\mathcal{U}) = m(\mathcal{U})$ for all $g \in G$ and $\mathcal{U} \subset G$.

- Amenable groups : finite groups, abelian groups, stable under extensions, subgroups, direct limits, ...
- Nonamenable groups : free groups $\mathbb{F}_n$, groups containing $\mathbb{F}_2$, ...
- Corollary of Connes' 1976 theorem: all $L(G)$ with G amenable and icc are isomorphic !

Amenable von Neumann algebras: full classification

Factor: a von Neumann algebra M with trivial center, i.e. $M \ncong M_1 \oplus M_2$.

$\leadsto$ $L(G)$ is a factor if and only if G is icc.

Type II_1 factor: admitting a faithful normal trace $\tau : M \to \mathbb{C}$.

Trace property: $\tau(xy) = \tau(yx)$.

$\leadsto$ $L(G)$ always has a trace, while $L^\infty(X) \rtimes G$ has a trace iff $G \curvearrowright X$ admits an invariant probability measure.

$\leadsto$ Modular theory of Tomita-Takesaki and Connes: reduction of arbitrary factors to II_1 factors.

Theorem (Connes, 1976)

All amenable II_1 factors are isomorphic, with the hyperfinite II_1 factor R defined by $M_2(\mathbb{C}) \subset M_4(\mathbb{C}) \subset M_8(\mathbb{C}) \subset \cdots \subset R$.

Beyond amenability: Popa's deformation/rigidity theory

Consider one of the most well studied group actions:

Bernoulli action $G \curvearrowright (X,\mu) = \prod_{g\in G}(X_0,\mu_0) : (g\cdot x)_h = x_{g^{-1}h}$.

- $M = L^\infty(X) \rtimes G$ is a II_1 factor.
- Whenever G is amenable, we have $M \cong R$.

Superrigidity theorem (Popa, Ioana, V)

If G has property (T), e.g. $G = \mathrm{SL}(n,\mathbb{Z})$ for $n \geq 3$,

or if $G = G_1 \times G_2$ is a non-amenable direct product group,

then $L^\infty(X) \rtimes G$ remembers the group G and its action $G \curvearrowright (X,\mu)$.

More precisely: if $L^\infty(X) \rtimes G \cong L^\infty(Y) \rtimes \Gamma$ for any other free, ergodic, probability measure preserving (pmp) group action $\Gamma \curvearrowright (Y,\eta)$,

then $G \cong \Gamma$ and the actions are conjugate (isomorphic).

Free groups

Theorem (Popa - V)

Whenever $n \neq m$, we have that $L^\infty(X) \rtimes \mathbb{F}_n \ncong L^\infty(Y) \rtimes \mathbb{F}_m$,

for arbitrary free, ergodic, pmp actions of the free groups.

- If $L^\infty(X) \rtimes \mathbb{F}_n \cong L^\infty(Y) \rtimes \mathbb{F}_m$, there also exists an isomorphism π such that $\pi(L^\infty(X)) = L^\infty(Y)$.

  This is thanks to **uniqueness of the Cartan subalgebra.**

- Such a π induces an **orbit equivalence:** a measurable bijection $\Delta : X \to Y$ such that $\Delta(\mathbb{F}_n \cdot x) = \mathbb{F}_m \cdot \Delta(x)$ for a.e. $x \in X$.

- (Gaboriau) The L^2-Betti numbers of a group are invariant under orbit equivalence.

  We have $\beta_1^{(2)}(\mathbb{F}_n) = n - 1$.

L^2-Betti numbers of groups

- Let G be a countable group. View $\ell^2(G)$ as a left G-module (by left translation) and a right $L(G)$-module (by right translation).
- **Atiyah, Cheeger-Gromov, Lück:** $\beta_n^{(2)}(G) = \dim_{L(G)} H^n(G, \ell^2(G))$.
- **Gaboriau:** invariant under orbit equivalence.

Conjecture (Popa, Ioana, Peterson)

If $L^\infty(X) \rtimes G \cong L^\infty(Y) \rtimes \Gamma$ for some free, ergodic, pmp actions, then $\beta_n^{(2)}(G) = \beta_n^{(2)}(\Gamma)$ for all $n \geq 0$.

$\rightsquigarrow$ (Popa-V) True if G is (among others) a hyperbolic group.

Big dream (many authors)

Define L^2-Betti numbers for II_1 factors. Prove $\beta_1^{(2)}(L(\mathbb{F}_n)) = n - 1$.

Key properties from harmonic analysis

- **Amenability:** there exist finite rank maps $\varphi_n : M \to M$ that are **completely positive** and that converge pointwise to the identity.

- **Weak amenability:** there exist finite rank maps $\varphi_n : M \to M$ that converge pointwise to the identity and satisfy $\sup_n \|\varphi_n\|_{\mathrm{cb}} < +\infty$.

⤳ Group von Neumann algebras of free groups (and of hyperbolic groups) are weakly amenable.

These are **deformation** (of the identity) properties.

⤳ Tension with a **rigidity** property like non-amenability, or even Kazhdan's property (T).

- **Property (T):** any sequence of completely positive maps $\varphi_n : M \to M$ converging pointwise to the identity must converge uniformly on the unit ball of M.