HOLOMORPHIC EXTENSIONS OF EIGENFUNCTIONS ON NA GROUPS

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(Based on joint work with Luz Roncal)

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In this setting, eigenfunctions of Δ_S are functions on $N \times A$. If $u(n, a)$, $n \in N$, $a \in A$, is an eigenfunction, we are interested in the holomorphic extendability of $n \to u(n, a)$, for a fixed.

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In a recent joint work with Luz Roncal we have shown that this is the case for certain eigenfunctions of the Laplace-Beltrami operator Δ_S.

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In the case of $\mathbf{H}$, the Iwasawa decomposition $G = NAK$ is given by $N = \mathbb{R}^n$, $K = SO(n)$ and $A = \mathbb{R}_+$. Thus we can identify $\mathbf{H}$ with the upper half-space $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+$ equipped with the Riemannian metric $g = \rho^{-2}(|dx|^2 + d\rho^2)$.

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We denote by Δ_g the Laplace-Beltrami operator associated to this metric. As this operator does not behave well with conformal change of metrics, we replace this with the new operator $L_g(w) = -\Delta_g w - \frac{n^2-1}{4} w$, which is conformally covariant.

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By letting $g_0 = \rho^2 g$ and making use of the conformal covariant property of L_g we calculate

$$L_g w = \rho^{\frac{n+3}{2}} L_{g_0}(\rho^{-\frac{n-1}{2}} w) = -\rho^{\frac{n-1}{2}} \rho^2 (\Delta + \partial_\rho^2)(\rho^{-\frac{n-1}{2}} w).$$

A simple calculation shows that

$$L_g = -\rho^2 \big(\Delta w + \partial_\rho^2\big) w + (n-1)\rho \partial_\rho w - \frac{n^2-1}{4} w.$$

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Real hyperbolic space

As $-\Delta_g = L_g + \frac{n^2-1}{4}$, the eigenfunction equation

$$-\Delta_g w = \gamma(n-\gamma)w, \quad \gamma = (s+n)/2$$

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The extension problem for the Laplacian

By the extension problem for the Laplacian Δ on $\mathbb{R}^n$ we mean the initial value problem, for $s > 0$,

$$\big(\Delta + \partial_\rho^2 + \frac{1-s}{\rho}\partial_\rho\big)u(x,\rho) = 0, \quad u(x,0) = f(x), \qquad x \in \mathbb{R}^n,\ \rho > 0.$$

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A solution of the above is explicitly given by $u(x,\rho) = \rho^s f * \varphi_{s,\rho}(x)$ where $\varphi_{s,\rho}$ is the generalised Poisson kernel

$$\varphi_{s,\rho}(x) = \pi^{-n/2}\frac{\Gamma(\frac{n+s}{2})}{|\Gamma(s)|}(\rho^2 + |x|^2)^{-\frac{n+s}{2}}.$$

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We observe that the generalised Poisson kernel has a holomorphic extension

$$\varphi_{s,\rho}(z) = \pi^{-n/2}\frac{\Gamma(\frac{n+s}{2})}{|\Gamma(s)|}(\rho^2 + z_1^2 + z_2^2 + ... + z_n^2)^{-\frac{n+s}{2}}. \tag{1}$$

to the tube domain

$$\Omega_\rho = \{z = x + iy \in \mathbb{C}^n : |y| < \rho\}. \tag{2}$$

Holomorphic extension of the Poisson integral

It then follows that for $f \in L^2(\mathbb{R}^n)$ the solution $u(x, \rho)$ also has a holomorphic extension $u(z, \rho)$ to Ω_ρ.

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The Fourier transforms of the Poisson kernels $\varphi_{s,\rho}$ are given by the Macdonald functions $K_{s/2}$. We let I_ν stand for Bessel functions of second kind and define a weight function

$$w_\rho(\xi) := c_{n,s}(\rho|\xi|)^s(K_{s/2}(\rho|\xi|))^2 \frac{I_{s+n/2-1}(2\rho|\xi|)}{(2\rho|\xi|)^{s+n/2-1}}. \qquad (3)$$

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We define $\mathcal{H}^s_\rho(\mathbb{R}^n)$ to be the Sobolev space of all tempered distributions f whose Fourier transforms $\widehat{f}$ are functions for which

$$\|f\|^2_{\mathcal{H}^s_\rho} := \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 w_\rho(\xi)\, d\xi < \infty.$$

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We let T_s stand for the operator taking f into the holomorphic extension of the solution $u(x,\rho) = \rho^s f * \varphi_{s,\rho}(x)$. We are interested in characterising the image of $\mathcal{H}^\rho_s$ under this map.

A weighted Bergman space

For any $s > 0$ we define the weighted Bergman space $\mathcal{B}_s(\Omega_\rho)$ consisting of all holomorphic functions F on Ω_ρ for which

$$\|F\|_{\mathcal{B}_s}^2 := \rho^{-n} \int_{\Omega_\rho} |F(x+iy,\rho)|^2 \Big(1 - \frac{|y|^2}{\rho^2}\Big)_+^{s-1} dx\, dy < \infty.$$

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Theorem 1

A solution of the extension problem is of the form $u(x,\rho) = f * \rho^s \varphi_{s,\rho}(x)$ for some $f \in L^2(\mathbb{R}^n)$ if and only if for each $\rho > 0$, $u(\cdot,\rho)$ extends to Ω_ρ as a holomorphic function, belongs to $\mathcal{B}_s(\Omega_\rho)$ and satisfies the uniform estimate $\|u(\cdot,\rho)\|_{\mathcal{B}_s} \leq C$ for all $\rho > 0$.

Return to the real hyperbolic space

For any $\lambda \in \mathbb{C}$ which is not a pole of $\Gamma(\frac{n-i\lambda}{2})$ we consider the kernels

$$\mathcal{P}_\lambda(x,\rho) = \pi^{-n/2}\frac{\Gamma(\frac{n-i\lambda}{2})}{\Gamma(-i\lambda)}\rho^{\frac{n-i\lambda}{2}}(\rho^2+|x|^2)^{-\frac{n-i\lambda}{2}},$$

which play the role of the Poisson kernels when **H** is identified with the group $S = NA$, $N = \mathbb{R}^n$, $A = \mathbb{R}_+$.

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Given a reasonable function f on $\mathbb{R}^n$ we define its Poisson transform by

$$\mathcal{P}_\lambda f(x,\rho) = \int_{\mathbb{R}^n}\mathcal{P}_\lambda(x-y,\rho)f(y)\mathcal{P}_{-\lambda}(y,1)du.$$

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In view the connection between eigenfunctions and solutions to the extension problem, the Poisson transform $\mathcal{P}_\lambda f$ is an eigenfunction of the hyperbolic Laplacian:

$$\Delta_g(\mathcal{P}_\lambda f) = -\frac{1}{4}(n^2+\lambda^2)\mathcal{P}_\lambda f$$

Complex hyperbolic spaces

We can now stae the following result on eigenfunctions of the hyperbolic Laplacian Δ_g as a corollary to Theorem 1.

Theorem 2

An eigenfunction $w(x,\rho)$ *of the Laplacian* Δ_g *with eigenvalue* $-\frac{1}{4}(n^2 - s^2)$ *is the Poisson integral* $\mathcal{P}_{is}f$ *with* $f \in L^2(\mathbb{R}^n, (1+|y|^2)^{-n+s}dy)$ *if and only if* $w(x,\rho)$ *extends to* Ω_ρ *as a holomorphic function, belongs to* $\mathcal{B}_s(\Omega_\rho)$ *and satisfies the estimate* $\|w(\cdot,\rho)\|_{\mathcal{B}_\rho} \leq C\rho^{(n-s)/2}$ *for all* $\rho > 0$.

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We consider eigenfunctions of the Laplace-Beltrami operator on the complex hyperbolic space $X = G/K$ where $G = SU(n+1,1)$ and $K = SU(n)$. Here $SU(n)$ is the special unitary group, that is, the Lie group of $n \times n$ unitary matrices with determinant 1.

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As in the case of real hyperbolic case we identify X with a solvable group $S = NA$ where $G = NAK$ is the Iwasawa decomposition.

The Heisenberg group and its Lie algebra

The Iwasawa decomposition of $G = SU(n+1,1)$ is explicitly given by $N = \mathbb{H}^n$, $K = SU(n)$ and $A = \mathbb{R}_+$. Here $\mathbb{H}^n$ is the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ equipped with the group law

$$(z,a)(z',a') = \left(z + z', a + a' + \frac{1}{2}\operatorname{Im}(z \cdot \bar{z}')\right), \tag{4}$$

where $z, z' \in \mathbb{C}^n$ and $a, a' \in \mathbb{R}$.

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It is convenient to use real coordinates: thus identifying $\mathbb{H}^n$ with $\mathbb{R}^{2n+1}$ and considering coordinates (x, u, ξ) we can write the group law as

$$(x,u,\xi)(y,v,\eta) = \big(x+y, u+v, \xi+\eta+\frac{1}{2}(u\cdot y - v\cdot x)\big).$$

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It is convenient to use real coordinates: thus identifying $\mathbb{H}^n$ with $\mathbb{R}^{2n+1}$ and considering coordinates (x, u, ξ) we can write the group law as

$$(x, u, \xi)(y, v, \eta) = \big(x + y, u + v, \xi + \eta + \frac{1}{2}(u \cdot y - v \cdot x)\big).$$

A a basis for the Heisenberg Lie algebra $\mathfrak{h}_n$ is give by the left invariant vector fields

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} u_j \frac{\partial}{\partial \xi}, \qquad Y_j = \frac{\partial}{\partial u_j} - \frac{1}{2} x_j \frac{\partial}{\partial \xi}, \qquad T = \frac{\partial}{\partial \xi}. \tag{5}$$

A solvable extension of $\mathbb{H}^n$.

The operator $\mathcal{L} = -\sum_{j=1}^{n} \left(X_j^2 + Y_j^2\right)$ is known as the sublaplacian and plays the role of the Laplacian on the Heisenberg group.

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As $\mathbb{R}_+$ acts on $\mathbb{H}^n$ as automorphisms, we can now form the semi-direct product of $\mathbb{H}^n$ with $\mathbb{R}^+$, $S = \mathbb{H}^n \times \mathbb{R}_+$ where the group law in S is given by

$$(z, a, \rho)(w, b, \rho') = \left(z + \sqrt{\rho} w, a + \rho b + \frac{1}{2}\sqrt{\rho}\,\mathrm{Im}(z \cdot \bar{w}), \rho\rho'\right).$$

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The Lie algebra $\mathfrak{s}$ of the Lie group S can be identified with $\mathbb{R}^{2n+1} \times \mathbb{R}$. And the vector fields

$$E_0 = \rho \partial_\rho,\ E_j = \sqrt{\rho} X_j,\ E_{n+j} = \sqrt{\rho} Y_j,\ E_{2n+1} = \rho T,$$

for $j = 1, 2, ..., n$ are left invariant on S.

The Laplace-Beltrami operator on S.

Equipping $\mathfrak{s}$ with the standard inner product on $\mathbb{R}^{2n+2}$ these $2n+2$ vector fields can be made to form an orthonormal basis for $\mathfrak{s}$. This induces a Riemannian metric on $\mathfrak{s}$.

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The Laplace-Beltrami operator on the Riemannian manifold S can be expressed in terms of the vector fields E_j. Indeed, we have

$$\Delta_S = \sum_{j=0}^{2n+1} E_j^2 - (n+1)E_0 = -\rho\mathcal{L} + \rho^2\partial_a^2 - (n+1)\rho\partial_\rho .$$

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As in the case of the hyperbolic Laplacian Δ_g on $\mathbb{R}^n \times \mathbb{R}_+$ we now consider the following eigenvalue problem on $S = NA$:

$$-\Delta_S \widetilde{W}(x,u,\xi,\rho) = \gamma(n+1-\gamma)\widetilde{W}(x,u,\xi,\rho), \quad \gamma = \frac{1}{2}(n+1+s), \quad s > 0.$$

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Eigenfunctions of Δ_S

We define W and U in terms of $\widetilde{W}$ as follows:

$$\widetilde{W}(x,u,\xi,\rho) = \rho^{\frac{n+1-s}{2}} W(x,u,\xi,\rho) = \rho^{\frac{n+1-s}{2}} U(2^{-1/2}(x,u), 2^{-1}\xi, \sqrt{2\rho}).$$

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Let $s > 0$. The function $\widetilde{W}$ is an eigenfunction of the Laplace-Beltrami operator Δ_S with eigenvalue $-\gamma(n+1-\gamma)$, $\gamma = \frac{1}{2}(n+1+s)$ if and only if the function U defined as

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Thus we are led to study the extension problem for the sublaplacian on the Heisenberg group.

The extension problem for the sublaplacian

By the extension problem for the sublaplacian $\mathcal{L}$ on $\mathbb{H}^n$ we mean the following initial value problem, for $s > 0$:

$$\big(-\mathcal{L} + \partial_\rho^2 + \frac{1-2s}{\rho}\partial_\rho + \frac{1}{4}\rho^2\partial_\xi^2 \big) U(x,u,\xi,\rho) = 0, \qquad U(x,u,\xi,0) = f(z,\xi). \tag{7}$$

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A solution of this problem is explicitly given by convolution with a kernel $\Phi_{s,\rho}$:

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A weighted Bergman space

More precisely, for $f \in L^2(\mathbb{H}^n)$ the solution U defined above holomorphically extends to the domain $\Omega_{\rho/4}$ where

$$\Omega_r = \Big\{(z, w, \zeta) \in \mathbb{C}^{2n+1} : \big|\, \mathrm{Im}(z, w, \zeta - \frac{1}{4}(z\bar{w} - w\bar{z}))\big| < r\Big\}. \tag{8}$$

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We observe that the domain Ω_r is invariant under the action of the Heisenberg motion group $G_n = \mathbb{H}^n \ltimes U(n)$ which is the semi-direct product of $\mathbb{H}^n$ and $U(n)$.

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We consider $\mathcal{O}(\mathbb{C}^{2n+1})$, the space of all holomorphic functions on $\mathbb{C}^{2n+1}$ and equip it with the L^2 norm

$$\|F\|^2 = \int_{\Omega_\rho} \big|F(z, w, \zeta)\big|^2 \, dz\, dw\, d\zeta. \tag{9}$$

We will denote by $\widetilde{\mathcal{H}}(\Omega_\rho)$ the completion of $\mathcal{O}(\mathbb{C}^{2n+1})$ with respect to the norm. It is easy to see that $\widetilde{\mathcal{H}}(\Omega_\rho) \subset \mathcal{O}(\Omega_\rho) \cap L^2(\Omega_\rho)$.

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A function space on $\mathbb{H}^n$

We introduce the following subspace of $L^2(\mathbb{H}^n)$. We let $\varphi_k^\lambda(z, w)$ stand for Laguerre functions of type $(n-1)$ and set

$$\Psi_k^\lambda(\rho) = \frac{k!(n-1)!}{(k+n-1)!} \int_{|(y,v,\eta)| \leq \rho} e^{2\lambda\eta} \varphi_k^\lambda(2iy, 2iv) dy\, dv\, d\eta. \tag{10}$$

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Theorem 3

Let $\rho > 0$. A function F from $L^2(\Omega_\rho)$ belongs to $\widetilde{\mathcal{H}}(\Omega_\rho)$ if and only if its restriction f belongs to $\mathcal{H}_\rho(\mathbb{H}^n)$. Moreover,

$$\int_{\Omega_\rho} |F(z,w,\zeta)|^2 \, dz\, dw\, d\zeta = C_n \int_{-\infty}^{\infty} \Big(\sum_{k=0}^{\infty} \|f^\lambda *_\lambda \varphi_k^\lambda\|_2^2 \, \Psi_k^\lambda(\rho) \Big) d\mu(\lambda).$$

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Gutzmer's formula for the Heisenberg motion group

In proving the above theorem we make use of the following result known as Gutzmer's formula.

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Let F be an entire function on $\mathbb{C}^{2n+1}$ and let f stand for the restriction of F to $\mathbb{H}^n$. Then we have the identity

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under the assumption that either the left hand side or the right hand side is finite.

We can now state the following result on the holomorphic extendability of solutions of the extension problem for the sublaplacian.

Holomorphic extendability of solutions of the extension problem

Theorem 5

For $0 < s \leq 1/2$*, let* $U(\cdot, \rho) = \rho^{2s} f * \Phi_{s,\rho}(\cdot)$ *where* $f \in L^2(\mathbb{H}^n)$*. Then for any* $0 < \gamma \leq -1 + \sqrt{2}$ *the solution of the extension problem* (7) *extends to* $\Omega_{\gamma\rho}$ *as a holomorphic function* $\widetilde{U}$*, belongs to* $\widetilde{\mathcal{H}}(\Omega_{\gamma\rho})$ *and satisfies the uniform estimate* $\|\widetilde{U}(\cdot, \rho)\|_{\widetilde{\mathcal{H}}(\Omega_{\gamma\rho})} \leq C\rho^{Q/2}\|f\|_2$ *for all* $\rho > 0$*.*

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In order to strengthen the above theorem and also to obtain a converse we introduce the space $\mathcal{H}^s_{\gamma,\rho}(\mathbb{H}^n)$ for $s, \gamma > 0$, as the completion of $C_0^\infty(\mathbb{H}^n)$ with respect to the norm

$$\|f\|^2_{\mathcal{H}^s_{\gamma,\rho}} = \int_{-\infty}^{\infty} \Big(\sum_{k=0}^{\infty} \|f^\lambda *_\lambda \varphi_k^\lambda\|_2^2 \; w_k^\lambda(\rho, \gamma\rho) \Big) d\mu(\lambda).$$

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where $w_k^\lambda(\rho, r) = \rho^{4s}(c_{k,\rho^2}^{\lambda/4}(s))^2 \psi_k^\lambda(r)$. Here $\psi_k^\lambda(r)$ are spherical functions on $\mathbb{H}^n$ and $c_{k,\rho^2}^\lambda(s)$ are given in terms of Kummer's hypergeometric functions.

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A Hardy space of homomorphic functions

We also introduce the following Hardy space on $\mathbb{H}^n$. For $s, \gamma > 0$, we define the space $\widetilde{H}^2(\Omega_{\gamma\rho})$ consisting of holomorphic functions on $\Omega_{\gamma\rho}$ for which

$$\|\widetilde{F}\|^2_{\widetilde{H}^2(\Omega_{\gamma\rho})} = \sup_{0<r<\gamma\rho} \int_{\mathbb{H}^n} \left(\int_{|b|=r} |\widetilde{F}(a+ib)|^2 d\sigma_r(b) \right) da < \infty.$$

With these notations, the proof of the Theorem 5 leads to the following result, which is the Heisenberg analogue of a Theorem proved for $\mathbb{R}^n$.

A Hardy space of homomorphic functions

We also introduce the following Hardy space on $\mathbb{H}^n$. For $s, \gamma > 0$, we define the space $\widetilde{H}^2(\Omega_{\gamma\rho})$ consisting of holomorphic functions on $\Omega_{\gamma\rho}$ for which

$$\|\widetilde{F}\|^2_{\widetilde{H}^2(\Omega_{\gamma\rho})} = \sup_{0<r<\gamma\rho} \int_{\mathbb{H}^n} \Big(\int_{|b|=r} |\widetilde{F}(a+ib)|^2 d\sigma_r(b) \Big) da < \infty.$$

With these notations, the proof of the Theorem 5 leads to the following result, which is the Heisenberg analogue of a Theorem proved for $\mathbb{R}^n$.

Theorem 6

For $0 < s \leq 1/2$, let $U(\cdot, \rho) = \rho^{2s} f * \Phi_{s,\rho}(\cdot)$ where $f \in \mathcal{H}^s_{\gamma,\rho}(\mathbb{H}^n)$. Then for any $0 < \gamma \leq -1 + \sqrt{2}$ the solution of the extension problem (7) extends to $\Omega_{\gamma\rho}$ as a holomorphic function $\widetilde{U}(\cdot, \rho)$, belongs to $\widetilde{H}^2(\Omega_{\gamma\rho})$ and satisfies the estimate $\|\widetilde{U}(\cdot, \rho)\|_{\widetilde{H}^2(\Omega_{\gamma\rho})} = C_s \|f\|_{\mathcal{H}^s_{\gamma,\rho}(\mathbb{H}^n)}$ for all $\rho > 0$. Moreover, the map $T_\rho : \mathcal{H}^s_{\gamma,\rho}(\mathbb{H}^n) \to \widetilde{H}^2(\Omega_{\gamma\rho})$ taking f into $\widetilde{U}(\cdot, \rho)$ is surjective.

Characterisations of eigenfunctions of Δ_S.

The above results lead to the following characterisations of solutions of the extension problem and eigenfunctions of Δ_S.

Theorem 7

A solution of the extension problem is of the form $U(x,u,\xi,\rho) = \rho^{2s} f * \Phi_{s,\rho}(x,u,\xi)$ *for some* $f \in L^2(\mathbb{H}^n)$ *if and only if there exists* $\gamma > 0$ *such that for each* $\rho > 0$, $U(\cdot,\rho)$ *extends to* $\Omega_{\gamma\rho}$ *as a holomorphic function* $\widetilde{U}(\cdot,\rho)$, *belongs to* $\widetilde{\mathcal{H}}(\Omega_{\gamma\rho})$ *and satisfies the uniform estimate* $\|\widetilde{U}(\cdot,\rho)\|_{\widetilde{\mathcal{H}}(\Omega_{\gamma\rho})} \leq C\rho^{Q/2}$ *for all* $\rho > 0$.

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Theorem 8

Let $0 < s < 1$. *An eigenfunction* $\widetilde{W}$ *of the Laplace-Beltrami operator* Δ_S *on* S *with eigenvalue* $-\frac{1}{4}((n+1)^2 - s^2)$ *can be expressed as the Poisson integral* $\mathcal{P}_{is}f$ *with* $f \in L^2(\mathbb{H}^n, (\varphi_{0,1}(h))^2 dh)$ *if and only if there exists a* $\gamma > 0$ *such that* $\widetilde{W}(\cdot,\rho) \in \widetilde{\mathcal{H}}(\Omega_{\gamma\sqrt{\rho}})$ *and satisfies the uniform estimate* $\|\widetilde{W}(\cdot,\rho)\|_{\widetilde{\mathcal{H}}(\Omega_{\gamma\sqrt{\rho}})} \leq C\rho^{(Q-s)/2}$ *for all* $\rho > 0$.

Thanks for your attention