Gabor frames and the Seshadri constant

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Basic notions – Time-frequency analysis

Given $f, g : \mathbb{R}^d \to \mathbb{C}$ and $z = (x, \omega) \in \mathbb{R}^{2d}$.

- *translation operator* T_x by $T_x g(t) = g(t - x)$,
- *modulation operator* M_ω by $M_\omega g(t) = e^{2\pi i \omega \cdot t} g(t)$
- *time-frequency shifts* $\pi(z)$ by $\pi(z) = M_\omega T_x$

Commutation relations

$$T_x M_\omega = e^{2\pi i x\omega} M_\omega T_x$$

$$\pi(z)\pi(z') = e^{2\pi i \sigma(z,z')} \pi(z')\pi(z),$$

where σ denotes the standard symplectic form on $\mathbb{R}^{2d}$,
$\sigma(z, z') = x' \cdot \omega - x \cdot \omega'$ for $z = (x, \omega)$ and $z' = (x', \omega')$.

Gabor systems

Given a lattice $L\mathbb{Z}^{2d}$ for $L \in \mathrm{GL}(2d, \mathbb{R})$, and a $g \in L^2(\mathbb{R}^d)$. A system of the form $\{\pi(Lk)g\}_{k\in\mathbb{Z}^{2d}}$ is called a **Gabor frame** if there exist constants $A, B > 0$ such that

$$A\|f\|_2^2 \leq \sum_{k\in\mathbb{Z}^{2d}} |\langle f, \pi(Lk)g\rangle|^2 \leq B\|f\|_2^2$$

for all $f \in L^2(\mathbb{R}^d)$.

Implication:
Existence of a dual atom $h \in L^2(\mathbb{R}^d)$ such that

$$f = \sum_{k\in\mathbb{Z}^{2d}} \langle f, \pi(Lk)g\rangle \pi(Lk)h.$$

for all $h \in L^2(\mathbb{R}^d)$.

Density Theorem

If $\{\pi(Lk)g\}_{k\in\mathbb{Z}^{2d}}$ is a Gabor frame, then $|\det(L)| \leq 1$.

Given a lattice $\Lambda = L\mathbb{Z}^{2d}$. Then we define the **symplectic dual lattice / adjoint lattice** by

$$\Lambda^\circ := \{z \in \mathbb{R}^{2d} : \; e^{2\pi i \sigma(Lk,z)} = 1 \; \text{ for all } \; k \in \mathbb{Z}^{2d}\}.$$

Example

For $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ we have $\Lambda^\circ = \beta^{-1}\mathbb{Z} \times \alpha^{-1}\mathbb{Z}$, where $\Lambda^\circ = L^\circ\mathbb{Z}^2$ for $L^\circ = J^T L^{-T} J$ and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Duality theorem

$\{\pi(Lk)g\}_{k\in\mathbb{Z}^{2d}}$ is a Gabor frame for $L^2(\mathbb{R}^d)$ if and only if $\{\pi(L^\circ k)g\}_{k\in\mathbb{Z}^{2d}}$ is a Riesz basic sequence.

Generalized Gaussians

$g_\Omega(t) := \overline{e^{\pi i t^T \Omega t}}$, where Ω lies in the Siegel upper half-space

$$\mathfrak{H} := \{\Omega \in \mathrm{GL}(2d, \mathbb{C}) : \Omega = \Omega^T, \ \ \mathrm{Im}\,\Omega \text{ is positive definite}\}.$$

is the class of Gabor atoms that we are focusing on.

Lyubarskii, Seip-Wallsten

$\{e^{-\pi t^2}, \alpha\mathbb{Z} \times \beta\mathbb{Z}\}$ is a Gabor frame **if and only if** $\alpha\beta < 1$.

Gaussian Gabor frames have received a lot of attention over the years, see for example contributions by Berndtsson, Ortega-Cerdà, Gröchenig, Lyubarskii, Lindholm,... .

Structure of Gabor systems

For $L = (\ell_1|\ell_2|\cdots|\ell_{2d})$ consider the commutation relations between the unitaries $\{\pi(\ell_j)|\, j = 1, ..., 2d\}$:

$$\pi(\ell_j)\pi(\ell_i) = e^{2\pi i\sigma(\ell_i,\ell_j)}\pi(\ell_i)\pi(\ell_j).$$

Note that $\Theta := (\sigma(\ell_i, \ell_j))_{i,j}$ is skew-symmetric and depends on $d(2d-1)$ parameters.
Hence for $d = 1$ we have one parameter θ, and for $d = 2$ there are 6 parameters: $\theta_{12}, \theta_{13}, \theta_{14}, \theta_{23}, \theta_{24}, \theta_{34}$.
For Gabor frames $\{\pi(Lk)g\}_{k\in\mathbb{Z}^2}$ there is just one parameter $\theta = \det L$ capturing the commutativity relations.
For Gabor frames $\{\pi(Lk)g\}_{k\in\mathbb{Z}^4}$ there is six parameters, none of them is $\det L$.

$$\begin{bmatrix} 0 & \theta_{12} & \theta_{13} & \theta_{14} \\ -\theta_{12} & 0 & \theta_{23} & \theta_{24} \\ -\theta_{13} & -\theta_{23} & 0 & \theta_{34} \\ -\theta_{14} & -\theta_{23} & -\theta_{34} & 0 \end{bmatrix}$$

Observe that the **Pfaffian** of Θ,

$$\mathrm{Pf}(\Theta) = \theta_{12}\theta_{34} - \theta_{13}\theta_{24} + \theta_{14}\theta_{23} = \det L,$$

since $\Theta = L^T J L$.
Equivalently,

$$\det L = \sigma(\ell_1, \ell_2)\sigma(\ell_3, \ell_4) - \sigma(\ell_1, \ell_3)\sigma(\ell_2, \ell_4) + \sigma(\ell_1, \ell_4)\sigma(\ell_2, \ell_3).$$

In other words, the covolume of L is solely expressed in parameters encoding the commutation relations of the lattice, which allows us to get an idea of the frame set for multi-variate Gabor frames

$$\{L \in \mathrm{GL}(2d, \mathbb{R}) : \ \pi(Lk)g\}_{k \in \mathbb{Z}^{2d}} \text{ is a frame.}\}$$

.

Shift of perspective

Θ-matrices encode the structure of Gabor lattices and not the generating matrices.

Observation

For $n = 2d$ we have that there exists an invertible $2d \times 2d$-matrix L such that $\Theta = L^T J L$ (symplectic Gramian of L), i.e. there is a lattice $L\mathbb{Z}^{2d}$ associated to a non-singular Θ.

We associate to Θ the skew-symmetric form $\sigma_\Theta(z, z') := \langle \Theta z, z' \rangle$.

Frame set

The frame set of a Gabor system $\{\pi(Lk)g\}_{k\in\mathbb{Z}^{2d}}\}$ is a subset of

$$\{(\theta_{ij}) \subseteq \mathbb{R}_{\geq 0}^{d(2d-1)} | \, \mathrm{Pf}(\Theta) \leq 1\}.$$

Theorem

Let Λ be a lattice in $\mathbb{R}^2$. Put

$$\Gamma := \{z \in \mathbb{C} : z = \eta + iy, \ \ (\eta, y) \in \Lambda^\circ\}, \ \ C := \frac{\pi}{4} \cdot \inf_{0 \neq \lambda \in \Gamma} |\lambda|^2.$$

Then $|\Gamma| = |\Lambda|^{-1} \geq C$ and we have the following estimate.

(1) If $C \geq 2$ then for all $f \in L^2(\mathbb{R})$ with $||f|| = 1$, we have

$$\frac{e}{4|\Lambda|} \leq \sqrt{2} \cdot \sum_{\lambda \in \Lambda} |(f, \pi_\lambda e^{-\pi t^2})|^2 \leq \frac{1}{(1 - e^{-C})|\Lambda|};$$

(2) If $1 < C < 2$ then for all $f \in L^2(\mathbb{R})$ with $\|f\|_2 = 1$, we have

$$\frac{(C-1)e}{C^2|\Lambda|} \leq \sqrt{2} \cdot \sum_{\lambda \in \Lambda} |(f, \pi_\lambda e^{-\pi t^2})|^2 \leq \frac{1}{(1 - e^{-C})|\Lambda|}.$$

Theorem-cont.

Assume further that $\Gamma = \mathrm{Span}_{\mathbb{Z}}\{1, \tau\}$ with $\mathrm{Im}\,\tau > 1$, then $C = \pi/4$ and for all $f \in L^2(\mathbb{R})$ with $\|f\|_2 = 1$, we have

$$\frac{4\pi(\mathrm{Im}\,\tau - 1)|\eta(\tau)|^6}{\left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 \mathrm{Im}\,\tau}\right)^2} \le \sqrt{2}\cdot\sum_{\lambda\in\Lambda} |(f, \pi_\lambda e^{-\pi t^2})|^2 \le \frac{\mathrm{Im}\,\tau}{1 - e^{-\pi/4}}, \qquad (1)$$

where $\eta(\tau) := e^{\pi i\tau/12}\Pi_{n=1}^{\infty}(1 - e^{2\pi i n\tau})$ is the Dedekind eta function.

Remarks

- The bounds in Eq. (1) are optimal and follow from the Ohsawa–Takegoshi extension theorem and Falting's work in Arakelov geometry.
- Since $|\eta(\tau)| > 0$ for all $\mathrm{Im}\,\tau > 1$ and $|\Gamma| = \mathrm{Im}\,\tau$, one may view (1) as an effective version of the behavior of frame bounds near critical density due to Borichev-Gröchenig-Lyubarskii.

General lattices

For general Γ, since $e^{-\pi|z|^2}$ is rotation invariant, one may assume that $\Gamma = \mathrm{Span}_{\mathbb{Z}}\{a, \tau\}$ with $a > 0$. But then the constant C will depend on a and τ and will not give the best frame bound.

Bergman kernel of the fundamental domain $D_\Gamma := \{ta + s\tau : -1/2 < t, s < 1/2\}$ of Γ

$$B_{D_\Gamma}(0) := \sup\{|f(0)|^2 : f \text{ is holom. on } D_\Gamma \text{ and } \int_{D_\Gamma} |f(z)|^2 e^{-\pi|z|^2} = 1\}.$$

Theorem

Let Λ be a lattice in $\mathbb{R}^2$. Assume that its symplectic dual lattice Γ in $\mathbb{C}$ is generated by $\{a, \tau\}$ with $a > 0$ and $a \operatorname{Im} \tau > 1$. Then for all $f \in L^2(\mathbb{R})$ with $\|f\|_2 = 1$, we have

$$\frac{4\pi(a \operatorname{Im}\tau - 1)|\eta(\tau/a)|^6}{\left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 \operatorname{Im}\tau/a}\right)^2} \le \sqrt{2} \cdot \sum_{\lambda\in\Lambda} |(f, \pi_\lambda e^{-\pi t^2})|^2 \le a \operatorname{Im}\tau \, B_{D_\Gamma}(0).$$

Key observation

Our starting point is a generalization of the sufficient part of Lyubarskii–Seip–Wallsten's result by Berndtsson–Ortega Cerdà based on the Hörmander L^2-estimate with singular weight for the $\overline{\partial}$-operator.

Our main contribution is that the Berndtsson–Ortega Cerdà approach also applies to the **higher-dimensional** case if one further introduces the notion of **Seshadri constant** into the picture.

Key step in Berndtsson–Ortega Cerdà

Another proof of the "sufficiency" part of the Lyubarskii–Seip–Wallsten theorem is based on the Hörmander $\overline{\partial}$ theory and implies the following identity:

$$|\Gamma| = \sup\{\gamma \geq 0 : \text{there exists a subharmonic function } \psi \text{ on } \mathbb{C} \text{ with isolated order } \gamma \log \text{ poles at } \Gamma \text{ and bounded from above by } \pi|z|^2\}, \tag{2}$$

where "ψ has isolated order γ log poles at Γ" means that ψ **is smooth outside** Γ **and there exist constants** $r, C > 0$ **such that**

$$\sup_{|z-\lambda|<r} |\gamma \log |z-\lambda|^2 - \psi(z)| < C, \qquad \forall\, \lambda \in \Gamma.$$

Main contribution

Our starting point is another proof of (2) by observing that the right hand side is precisely the **Seshadri constant** $\epsilon(\omega, X; 0)$ of the Euclidean Kähler metric $\omega := \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$ at the identity element, say 0, of the torus X in the one dimensional case.

For general algebraic varieties, Demailly gave several equivalent definition of the **Seshadri constant.** The one suitable for the compact Kähler case is

$$\epsilon(\omega, X; 0) := \sup\{\gamma \geq 0 : \text{there exists a } 2\pi\omega \text{ plurisubharmonic function on } X \text{ with isolated order } \gamma \text{ log poles at } 0\},$$

where a function ψ is said to be $2\pi\omega$ **plurisubharmonic** on X if ψ is upper semi-continuous and $i\partial\bar{\partial}\psi + 2\pi\omega \geq 0$ on X.

cont.

In the one dimensional case, the Hodge decomposition directly gives

$$\epsilon(\omega, X; 0) = |\Gamma|,$$

which gives another proof of the key step in Berndtsson-Ortega Cerda, (2).

Kähler geometry in a nutshell

For $z_1 = x_1 + i\omega_1$ and $z_2 = x_2 + i\omega_2$ we have

$$z_1\overline{z_2} = x_1\omega_1 + x_2\omega_2 + i(x_2\omega_1 - x_1\omega_2)$$

replace the standard Euclidean product by a Riemannian metric... .

Link to complex analysis

Bargman connection

Ω-**Bargman transform**

$$\mathcal{B}_\Omega f(z) := \int_{\mathbb{R}^n} f(t) e^{\pi i t^T \Omega t} e^{-2\pi i z^T t} dt_1 \cdots dt_n$$

and **Bargman-Fock space** for $z = \xi + \Omega x$ and $n = 2d$.

$$\mathcal{F}^2_\Omega := \{F \in \mathcal{O}(\mathbb{C}^n) : \int_{\mathbb{C}^n} |F(z)|^2 e^{-2\pi |\mathrm{Im} \Omega z|^2} d\xi_1 \cdots d\xi_n dx_1 \cdots dx_n < \infty\}$$

Sets of interpolation

Suppose we have a set $\Lambda = \{\lambda_j\}_{j \in J}$ such that for every $(a_j)_{j \in J} \in \ell^2$ there exist a constant $C > 0$ $f \in \mathcal{F}^2_\Omega$ such that $f(\lambda_j) = a_j$ and $\|f\|_{\mathcal{F}^2_\Omega} \leq C$ for $\sum_{j \in J} |a_j|^2 e^{-2\pi |(\mathrm{Im} \Omega) \lambda_j|^2} = 1$. Then Λ is said to be a **set of interpolation** (with bound C).

Equivalence

$\{\pi(Lk)g_\Omega\}_{k\in\mathbb{Z}^{2d}}$ is a Gabor frame for $L^2(\mathbb{R}^d)$
if and only if $\Gamma := \{\xi + \Omega x \in \mathbb{C}^n : (\xi, x) \in \Lambda^\circ\}$ is a set of interpolation for $\mathcal{F}^2_\Omega$ if and only if

$$\Gamma_{\Omega,\Lambda^\circ} := \{(\mathrm{Im}\,\Omega)^{-1/2} z \in \mathbb{C}^n : z = \eta + \Omega y, \quad (\eta, y) \in \Lambda^\circ\}$$

is a set of interpolation for the Bargmann–Fock space $\mathcal{F}^2$, where $(\mathrm{Im}\,\Omega)^{-1/2}$ denotes the unique positive definite matrix whose square equals to $(\mathrm{Im}\,\Omega)^{-1}$.

Theorem

The following are equivalent:

(1) $\Gamma_{\Omega,\Lambda^\circ}$ is a set of interpolation for $\mathcal{F}^2$ and for all $F \in \mathcal{F}^2$, $\sum_{\gamma\in\Gamma_{\Omega,\Lambda^\circ}} |F(\gamma)|^2 e^{-\pi|\gamma|^2} = 1$,

$$A \leq \inf_{F'\in\mathcal{F}^2,\, F'=F \text{ on } \Gamma_{\Omega,\Lambda^\circ}} ||F'||^2 \leq B;$$

(2) (Λ, g_Ω) defines a frame in $L^2(\mathbb{R}^d)$ and for all $f \in L^2(\mathbb{R}^d)$, $||f|| = 1$,

$$\frac{(B\cdot|\Lambda|)^{-1}}{\sqrt{2^n \det(\operatorname{Im}\Omega)^3}} \leq \sum_{\lambda\in\Lambda} |(f, \pi_\lambda g_\Omega)|^2 \leq \frac{(A\cdot|\Lambda|)^{-1}}{\sqrt{2^n \det(\operatorname{Im}\Omega)^3}}.$$

Theorem A

If $\epsilon(\omega, X; 0) > n$, then Γ is a set of interpolation for $\mathcal{F}^2$.

Theorem B

If $\inf_{0 \neq \lambda \in \Gamma} |\lambda|^2 > \frac{4n}{\pi}$, then Γ is a set of interpolation for $\mathcal{F}^2$. Assume further that

$$\inf_{0 \neq \lambda \in \Gamma} |\lambda|^2 \geq \frac{4(n+1)}{\pi},$$

with constant $C = (n+1)^{n+1} e^{-n}/n!$.

Theorem C

Let $\Gamma = \mathrm{Span}_{\mathbb{Z}}\{1, \tau\}$ be a lattice in $\mathbb{C}$. Then $\epsilon(\omega, X; 0) = |\Gamma| = \mathrm{Im}\,\tau$. Assume further that $\mathrm{Im}\,\tau > 1$. Then Γ is a set of interpolation for $\mathcal{F}^2$ with an interpolation bound

$$C = \frac{\mathrm{Im}\,\tau}{\mathrm{Im}\,\tau - 1} \cdot \frac{\left(\sum_{n \in \mathbb{Z}} e^{-\pi n^2 \mathrm{Im}\,\tau}\right)^2}{4\pi \cdot |\eta(\tau)|^6},$$

By Tosatti's formula for the Sehsadri constant we have that if the only positive dimensional irreducible analytic subvariety of X is X itself then $\epsilon(\omega, X; 0)^n/n! = |\Gamma|$.

Theorem D

(1) If Γ is a set of interpolation for $\mathcal{F}^2$ and all irreducible analytic subvarieties of X are translates of complex tori, then $\epsilon(\omega, X; 0) > 1$;

(2) If Γ is a set of interpolation for $\mathcal{F}^2$ and the only positive dimensional irreducible analytic subvariety of X is X itself then $\epsilon(\omega, X; 0)^n/n! = |\Gamma| > 1$.

Example

There exists a lattice (constructed in a recent paper by Gröchenig-Lyubarskii) in $\mathbb{C}^2$ whose Seshadri constant is bigger than one but it is not a set of interpolation for $\mathcal{F}^2$.

Let Γ be a lattice in $\mathbb{C}^n$. We call $m(\Gamma) := \inf_{0 \neq \mu \in \Gamma} |\mu|^2$ the **Buser–Sarnak invariant** of Γ.

Lazarsfeld's Theorem

$\epsilon(\omega, X; 0) \geq \frac{\pi}{4} m(\Gamma)$

A lattice Γ is called a **complex lattice** if $i\Gamma = \Gamma$.

Characterization

For a lattice Γ in $\mathbb{C}^n$, the followings are equivalent

(1) Γ is a complex lattice;

(2) $\Gamma = A\mathbb{Z}[i]^n$ for some $A \in GL(n, \mathbb{C})$;

(4) $X := \mathbb{C}^n/\Gamma$ is biholomorphic to $\mathbb{C}^n/\mathbb{Z}[i]^n$.

Estimate

Assume that $\Gamma = A\mathbb{Z}[i]^n$ is a complex lattice. Then

$$m(\Gamma) \geq \epsilon(\omega, X; 0) \geq \max\left\{\frac{\pi}{4} m(\Gamma),\ e_{min}(A)\right\},$$

where

$$e_{min}(A) := \inf_{z \in \mathbb{C}^n,\, |z|=1} |Az|^2.$$

Example

In case

$$\Gamma = a_1\mathbb{Z}[i] \times \cdots \times a_n\mathbb{Z}[i],\quad a_j > 0,$$

we have

$$e_{min}(A) = m(\Gamma) = \min\{a_1^2, \cdots, a_n^2\},$$

thus the above theorem gives

$$\epsilon(\omega, X; 0) = \min\{a_1^2, \cdots, a_n^2\}.$$

Sehsadri admissibility

Let

$$\{0\} = X_0 \subset X_1 \cdots \subset X_k = X := \mathbb{C}^n/\Gamma, \quad n_k := \dim_{\mathbb{C}} X_k, \quad k \geq 1,$$

be an increasing sequence of complex Lie subgroups of X. We shall introduce the Seshadri constant ϵ_j, $1 \leq j \leq k$, for extension from X_{j-1} to X_j. Let $\pi_j : E_j \to X_j$ be the covering map, where E_j is an n_j dimensional complex subspace of $\mathbb{C}^n$. Let $E_j = E_{j-1} \oplus F_j$, be the orthogonal decomposition with respect to the Euclidean metric ω. Then $\Gamma_j := F_j \cap \pi_j^{-1}(X_{j-1})$ defines a lattice in F_j. Put $X_{j-1}^{\perp} := F_j/\Gamma_j$, (in general, $X_{j-1}^{\perp}$ is not a subtorus of X_j). Denote by

$$\epsilon_j := \epsilon(\omega, X_{j-1}^{\perp}; 0)$$

the **Seshadri constant** at the origin of $X_{j-1}^{\perp}$ with respect to ω.

Sehsadri admissibility-continued

We call (23) an admissible sequence of X if

$$\epsilon_j > n_j - n_{j-1}, \ \ \forall\, 1 \leq j \leq n.$$

X is said to be **Seshadri admissible** if it possesses an admissible sequence.

Theorem

Assume that X is Seshadri admissible, then Γ is a set of interpolation in $\mathcal{F}^2$.

Complex lattices–Gröchenig

Let $\Gamma_{\Omega,\Lambda^\circ}$ be a complex lattice. With the notation above, assume that $\lambda_j > 1$ for all $1 \leq j \leq n$. Then $\Gamma_{\Omega,\Lambda^\circ}$ is set of interpolation for $\mathcal{F}^2$ (and equivalently $\{\pi(\lambda)g_\Omega\}_{\lambda\in\Lambda}$ defines a frame in $L^2(\mathbb{R}^d)$).