Riesz distributions and the Wallach set in rational Dunkl theory

Margit Rösler (Paderborn)

Noncommutative Conference
August 18-20, 2020
Ghent University, on Zoom

Classical: Riesz distributions on Hermitian matrix spaces

- **Hermitian $n \times n$ matrices over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ (or $\mathbb{H}$):**

$$H := H_n(\mathbb{F}) := \{x \in M_n(\mathbb{F}) : x = x^*\}; \; x^* = \overline{x}^t$$

  Euclidean vector space with $\langle x, y \rangle = \mathrm{Re}\,\mathrm{tr}(xy)$.

- **Positive definite matrices**:

$$\Omega := \Omega_n(\mathbb{F}) := \{x \in H_n(\mathbb{F}) : x \text{ positive definite}\}$$

Riesz measures on H:

Let $\mu \in \mathbb{C}$ with $\mathrm{Re}\,\mu > \mu_0 := \frac{d}{2}(n-1)$, $d := \dim_{\mathbb{R}} \mathbb{F} \in \{1, 2, 4\}$.
Define a complex measure R_μ on H by

$$\langle R_\mu, \varphi \rangle = \frac{1}{\Gamma_\Omega(\mu)} \int_\Omega \det(x)^{\mu - \mu_0 - 1} \varphi(x) dx, \quad \varphi \in C_c(H).$$

$$\Gamma_\Omega(\mu) = \int_\Omega e^{-\mathrm{tr}\, x}\, det(x)^{\mu - \mu_0 - 1} dx : \; \textbf{Gamma function of } \Omega$$

. **Notice:** We may also consider R_μ as a tempered distribution on H.

Analytic continuation with respect to μ

- $\det\left(\frac{\partial}{\partial x}\right) R_\mu = R_{\mu-1}$ in $\mathcal{S}'(H)$.
- Laplace transform:

$$\mathcal{L}R_\mu(y) := \int_\Omega e^{-\langle x,y\rangle} dR_\mu(x) = \det(y)^{-\mu}\,; \quad y \in \Omega.$$

Consequences:

- $\mu \mapsto R_\mu$ extends from $\{\mathrm{Re}\,\mu > \mu_0\}$ to an analytic mapping on $\mathbb{C}$ with values in $\mathcal{S}'(H)$, i.e. $\mu \mapsto \langle R_\mu, \varphi\rangle$ is analytic on $\mathbb{C}$ for all $\varphi \in \mathcal{S}(H)$.

  $R_\mu,\ \mu \in \mathbb{C}$: **Riesz distributions** associated with Ω.
- The Laplace transform formula extends to all $\mu \in \mathbb{C}$. (Important for the study of the R_μ!)

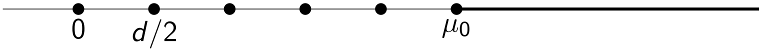
Applications:

- Multivariate statistics: Wishart distributions
- Representation theory. Important in this context: For which μ is R_μ a positive measure?

Theorem (Gindikin, 1975)

The Riesz distribution R_μ is a positive measure exactly if μ belongs to the **Wallach set**

$$\Big\{0, \frac{d}{2}, \ldots, \frac{d}{2}(n-1) = \mu_0\Big\} \cup \big\{\mu \in \mathbb{R} : \mu > \mu_0\big\}.$$



Important: The discrete points of the Wallach set are poles of Γ_Ω

Observation by A. Sokal (2011): If R_μ is a measure on H, then

$$\frac{1}{\Gamma_\Omega(\mu)} \det(x)^{\mu-\mu_0-1} \cdot \mathbb{1}_\Omega(x) \overset{!}{\in} L^1_{loc}(H)$$

- **This implies:** Either $\operatorname{Re}\mu > \mu_0$, or μ is a pole of Γ_Ω.
- **But:** For the discrete Wallach points, R_μ is **no longer** given by the above density, but supported in $\partial\Omega$!
- Sokal's observation considerably simplifies the proof by Gindikin.

A bridge to Dunkl theory

Consider the Riesz measures R_μ with $\operatorname{Re}\mu > \mu_0$ on H.

Observation:

Let $y \in \Omega$ with eigenvalues $\eta_1, \dots, \eta_n \in]0, \infty[=: \mathbb{R}_+$. Then

$$\mathcal{L}R_\mu(y) = \frac{1}{\Gamma_\Omega(\mu)} \int_\Omega e^{-\langle x,y\rangle} \det(x)^{\mu-\mu_0-1} dx$$
$$= const \cdot \int_{\mathbb{R}_+^n} J_{d/2}(-\xi, \eta)\, \Delta(\xi)^{\mu-\mu_0-1} \prod_{1\leq i<j\leq n} |\xi_i - \xi_j|^d d\xi$$

with $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}_+^n$.

- $\Delta(\xi) = \prod_{i=1}^n \xi_i$
- J_k : **Dunkl-type Bessel function** associated with root system A_{n-1} and multiplicity parameter $k \geq 0$.

Dunkl theory

- generalizes important aspects of harmonic analysis on Riemannian symmetric spaces
- fundamental ingredient: Dunkl operators = differential reflection operators associated with root systems
- **Here:** "rational" theory (Dunkl 1989): the Dunkl operators have rational coefficients

Setting:

- $R \subset \mathbb{R}^n$ a (not necessarily crystallographic) root system
- $W = \langle \sigma_\alpha : \alpha \in R \rangle$ associated finite reflection group (Weyl group)
- $k : R \to [0, \infty[\,, \alpha \mapsto k_\alpha$ a W-invariant **multiplicity function**

Example: $R = A_{n-1} = \{\pm(e_i - e_j) : 1 \le i < j \le n\} \subset \mathbb{R}^n$

- $W = S_n$ (acts by permutation of the coordinates),
- 1 multiplicity parameter $k \in [0, \infty[$.

Dunkl operators $T_\xi = T_\xi(k)$ associated with R and k:

$$T_\xi f(x) = \partial_\xi f(x) + \frac{1}{2}\sum_{\alpha\in R} k_\alpha \langle \alpha, \xi\rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle} \quad (\xi \in \mathbb{R}^n)$$

- The T_ξ, $\xi \in \mathbb{R}^n$ **commute!** (Dunkl 89)
- Nice mapping properties. In particular: T_ξ acts continuously on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and on $\mathcal{S}'(\mathbb{R}^n)$ via $\langle T_\xi u, \varphi\rangle := -\langle u, T_\xi \varphi\rangle$.

Commutative algebra of Dunkl operators:

$$\{p(T) : p \in \mathbb{C}[\mathbb{R}^n]\} \quad (\text{replace } x_i \text{ by } T_{e_i})$$

Theorem (Dunkl, Opdam)

There is a unique analytic function $E = E_k$ on $\mathbb{C}^n \times \mathbb{C}^n$ (the Dunkl kernel) with

$$T_\xi E(\,.\,, y) = \langle y, \xi\rangle E(\,.\,, y), \;\; E(0, y) = 1 \quad \forall\, y \in \mathbb{C}^n, \xi \in \mathbb{R}^n.$$

Case $k = 0$**:** $E(x, y) = e^{\langle x, y\rangle}$ $\qquad$ ($\langle .,. \rangle$ bilinear)

Basic properties:

- $E(x,y) = E(y,x)$
- $E(\lambda x, y) = E(x, \lambda y),\ E(wx, wy) = E(x,y) \quad \forall \lambda \in \mathbb{C}, w \in W$

Bessel function associated with R and k:

$$J(x,y) = \frac{1}{|W|!} \sum_{w \in W} E(wx, y) \quad (W\text{-invariant in } x, y)$$

For crystallographic R and special values of k, the $J(\,.\,, y)$ can be identified with the spherical functions of flat symmetic spaces.

Examples:

- **rank-one case:** $R = \{\pm 1\} \subseteq \mathbb{R};\ W = \{id, \sigma\}, \sigma(x) = -x$

  $$J(x,y) = j_{k-1/2}(ixy) \quad \text{(normalized 1-variable Bessel function)}$$

- $R = A_{n-1}$: J has an explicit expansion in terms of Jack polynomials, which depend on k and generalize the Schur polynomials ($k = 1$).

The Dunkl transform

Weight function: $\omega(x) := \prod_{\alpha \in R} |\langle \alpha, x \rangle|^{k_\alpha}$

- For $f \in L^1(\mathbb{R}^n, \omega)$,

$$\widehat{f}(y) := \int_{\mathbb{R}^n} f(x) E(x, -iy) \omega(x) dx, \quad y \in \mathbb{R}^n$$

(known: $|E(x, iy)| \leq 1$ for all $x, y \in \mathbb{R}^n$)

- **Rich harmonic analysis, close to classical Fourier analysis!**

Laplace transform?

Problem: in general, no nice decay properties available for $x \mapsto E(-x, y)$ as $x \to \infty$ within some convex cone $C \subset \mathbb{R}^n$ (and $y \in C$).

Laplace transform in the type A Dunkl setting

- Always: $R = A_{n-1}$ in $\mathbb{R}^n$, multiplicity $k \geq 0$.
- weight function: $\omega(x) = \prod_{1\leq i<j\leq n} |x_i - x_j|^{2k}$
- Consider the cone $\mathbb{R}_+^n$, $\mathbb{R}_+ =]0,\infty[$.

E **has good decay properties** (R., 2020):
Let $z \in \mathbb{C}^n$ with $\mathrm{Re}\, z \geq \mathbf{s} = (s,\ldots,s) \in \mathbb{R}^n$ (componentwise). Then

$$|E(-x,z)| \leq e^{-\langle x,\mathbf{s}\rangle} \quad \forall\, x \in \mathbb{R}_+^n.$$

Laplace transform of functions: For $f \in L^1_{loc}(\mathbb{R}_+^n)$,

$$\mathcal{L}f(z) := \int_{\mathbb{R}_+^n} E(-x,z)f(x)\,\omega(x)dx,\ z \in \mathbb{C}^n \quad \text{(if convergent)}$$

- If $|f(x)| \leq Ce^{\langle x,\mathbf{s}\rangle}$, then $\mathcal{L}f$ is holomorphic on $\{z \in \mathbb{C}^n : \mathrm{Re}\, z > \mathbf{s}\}$.
- $\exists$ Cauchy-type inversion theorem, standard injectivity results

History

- first introduced by I.G. Macdonald (unpublished manuscript 1987/88; arXiv:1309.4568), but only formally, and with the Bessel function J (before Dunkl theory!)
- used by several further authors (e.g. Baker/Forrester 1998: Calogero-Moser models) **But: must results not rigorous**

Laplace transform of tempered distributions:

The Laplace transform can be extended to tempered distributions on $\mathbb{R}^n$ which are supported in $\overline{\mathbb{R}^n_+}$. It is also injective.

Riesz distributions in the type A Dunkl setting

Always: $R = A_{n-1}$, multiplicity $k > 0$, $\mu_0 := k(n-1)$

Riesz measures on $\mathbb{R}^n$: For $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > \mu_0$,

$$\langle R_\mu, \varphi\rangle := \frac{1}{\Gamma_M(\mu)} \int_{\mathbb{R}^n_+} \varphi(x)\Delta(x)^{\mu-\mu_0-1}\omega(x)dx; \quad \varphi \in \mathcal{S}(\mathbb{R}^n)$$

- $\Delta(x) = x_1 \cdot \ldots \cdot x_n$
- $\Gamma_M(z) = c \cdot \prod_{j=1}^{n} \Gamma(z - k(j-1))$ **(Macdonald gamma function)**

Theorem (Y. Liu 2016, unpublished PhD thesis)

The mapping $\mu \mapsto R_\mu$ extends to an analytic function on $\mathbb{C}$ with values in $\mathcal{S}'(\mathbb{R}^n)$, satisfying

$$\Delta(T)R_\mu = R_{\mu-1}.$$

Further Properties of the Riesz distributions (M.R., 2020)

Theorem 1

(1) $R_\mu \in \mathcal{S}'(\mathbb{R}^n)$ is S_n-invariant and supported in $\overline{\mathbb{R}^n_+}$.

(2) **Laplace transform:**

$$\mathcal{L}R_\mu(y) = \Delta(y)^{-\mu} \quad \forall\, y \in \mathbb{R}^n_+$$

(3) $R_0 = \delta_0$ (because $\mathcal{L}R_0 = 1 = \mathcal{L}\delta_0$).

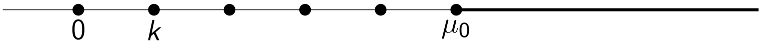
Proof of (2): based on hypergeometric expansion of the Bessel function J in terms of Jack polynomials and integral formulas of Macdonald.

Question: Which of the R_μ are positive measures?

The analogue of Gindikin's result

Generalized Wallach set:

$$W = W_k := \{0, k, \ldots, k(n-1) = \mu_0\} \cup \{\mu \in \mathbb{R} : \mu > \mu_0\}.$$



Theorem 2

(1) For $j = 0 \ldots, n-1$, the Riesz distributions R_{kj} on $\mathbb{R}^n$ are positive measures. They can be written down explicitly by recursion.

(2) $supp(R_{kj}) = \{x \in \partial(\mathbb{R}^n_+) :$ exactly j components of x are $\neq 0\}$.

(3) R_μ is a positive measure $\iff \mu \in W$

Proof: (1) + (2): Guess the candidate for R_{kj} and show by calculation that its Dunkl-Laplace transform coincides with that of R_{kj}.

(3) R_μ is positive measure $\Longrightarrow \mu \in W$:

- $\mu \geq 0$ by growth of $\mathcal{L}R_\mu$.
- **Sokal's method** gives: If R_μ is a measure, then either $\mu > \mu_0$, or $\mu \in \{0, k, \dots, k(n-1)\} - \mathbb{N}_0$ (poles of Γ_M).
- Exclude $\mu \in]k(j-1), kj[\,,\ 1 \leq j \leq n-1$:

  Suppose R_μ is a positive measure. Then for each polynomial p with $p \geq 0$ on $\overline{\mathbb{R}^n_+}$,

$$(*)\ \ p(-T)(\mathcal{L}R_\mu)(y) = \int_{\overline{\mathbb{R}^n_+}} p(x) \underbrace{E(-x,y)}_{\geq 0} dR_\mu(x) \ \geq 0 \ \ \forall y \in \mathbb{R}^n_+.$$

  (Variant of the **Shanbhag principle**).

  Take $p = e_{j+1}$ (elementary symmetric polynomial). Then $(*)$ is not satisfied.

Further relevance of the generalized Wallach set

Dunkl's intertwining operator: For each root system $R \subset \mathbb{R}^n$ and multiplicity $k \geq 0$, there exists a unique linear isomorphism V_k of $\mathbb{C}[\mathbb{R}^n]$ which preserves the degree of homogeneity and satisfies

$$V_k(1) = 1, \quad T_\xi(k)V_k = V_k\partial_\xi \quad \text{for all } \xi \in \mathbb{R}^n.$$

Known (M.R. 1999): V_k **is positive,** i.e. $V_k p \geq 0$ if $p \geq 0$.

For multiplicities k, k' on R, $T_\xi(k')(V_{k'} \circ V_k^{-1}) = (V_{k'} \circ V_k^{-1})T_\xi(k)$.

Old conjecture (P): If $k' \geq k$, then $V_{k'} \circ V_k^{-1}$ is positive.

Equivalent: There exist compactly supported probability measures $\mu_x^{k,k'}$ on $\mathbb{R}^n$ such that

$$E_{k'}(x,z) = \int_{\mathbb{R}^n} E_k(\xi, z)d\mu_x^{k,k'}(\xi) \quad \forall z \in \mathbb{C}^n \quad \text{(Sonine formula)}$$

(P) is true in rank 1 (Y. Xu 2003)

Results on conjecture (P) (with M. Voit, 2020)

Here:

- root system $R = B_n = \{\pm e_i, \pm e_i \pm e_j, 1 \leq i < j \leq n\} \subset \mathbb{R}^n$
- $k = (k_1, k_2)$ with $k_1 \geq 0$ (on $\pm e_i$), $k_2 > 0$ (on $\pm e_i \pm e_j$),
- $k' := (k_1 + h, k_2)$ with $h \geq 0$.

Theorem

(1) **Necessary condition:** *If $V_{k'} \circ V_k^{-1}$ is positive, then* **either** $h > k_2(n-1)$ **or** $h \in \big\{0, k_2, \ldots, k_2(n-1)\big\} - \mathbb{N}_0$.

(2) **Sufficient condition:** *If h belongs to the generalized Wallach set*

$$W_{k_2} = \big\{0, k_2, \ldots, k_2(n-1)\big\} \cup \,]k_2(n-1), \infty[\,,$$

then $V_{k'} \circ V_k^{-1}$ is positive on B_n-invariant polynomials.

References:

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Thank you for your attention!